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Optimal inventory strategies for EOQ-type reverse logistics systems

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Abstract

Deterministic EOQ-type two-store reverse logistics systems was studied extensively in the literature. All this proposition assumed a predetermined control system and searched the optimal parameters for this control systems. There are no results regarding the structure of optimal policies. The aim of the paper is to find the optimal inventory policies in these systems. It will be shown that an optimal policy can be characterized by a common meta-model.

Keywords: Remanufacturing; Reverse Logistics; EOQ; Lotsizing; Optimal Policies

Összefoglalás

Determinisztikus kétraktáras reverz logisztikai tételnagyság modelleket kiterjedten vizsgálták az irodalomban. A modellek nagy része előre meghatározott irányítási rendszert vizsgált és az ehhez tartozó optimális paramétereket határozza meg. Ismereteink szerint nem vizsgálták az optimális készletezési stratégiát. A dolgozat célja, hogy optimális készletezési stratégiát keressen ezekhez a rendszerekhez. Megmutatjuk, hogy az optimális stratégia egy általános meta-modellel jellemezhető.

Kulcsszavak: Újrafeldolgozás; Reverz Logisztika; Tételnagyság; Optimális Stratégia

1. Introduction

Reverse logistics is a term for manufacturing of materials and remanufacturing of from market returned reusable materials. The demand is to be satisfied with new manufactured (produced) and the remanufactured products, so there is no difference between manufactured and remanufactured items. In order the environment with waste not to burden, or the use of minerals to spare, it is a good opportunity to reuse the used items, if it is economical.

Quantitative models for inventory systems with remanufacturing provide a relevant generalisation of classical EOQ models. They extend the classical production-inventory models by introducing the opportunity of covering certain part of the demand by items returned by customers after use, which then, after some remanufacturing, can be reused instead of new ones.

EOQ-type reverse logistics models were studied extensively in the literature [1-14]. In this models the manufacturing and remanufacturing activities succeed each other, and the measure of effectiveness of such systems is the sum of setup and linear inventory holding costs. The main question is the number of the manufacturing and remanufacturing batches. All these approaches assumed a predetermined control system and searched the optimal parameters for this control systems. In the predetermined control policies it is assumed that all manufacturing lot sizes are equal and the remanufacturing lots, as well. There are no results regarding the structure of optimal policies [15].

The author has dealt with three of these proposals, has compared these models [1, 4, 7] and he has shown their mathematical equivalence to a model proposed earlier by the author [13].

The word *cycle* is here used to express that we are considering only such policies, where some fixed sequence of manufacturing and remanufacturing batches is repeated continuously.

The aim of the paper is to investigate the optimal remanufucturing and manufacturing policy in model of [1]. We ask, whether the manufacturing and remanufacturing lot sizes are equal that there is a common assumption in some of the model [1, 4], or not. At first, the minimal inventory holding costs are determined in a cycle under the condition that numbers of remanufacturing and manufacturing batches are fixed. Secondly, the optimal cycle time is determined. Analyzing the optimal number of remanufacturing and batches, it will be used the meta-model offered in [13]. To solve the problem, we apply a total cost approach instead of the traditional average cost method to show that both of the method lead to the same result. The advantage of the average cost is that it is enough to calculate the costs in a cycle and it is not necessary to determine all cosots in the planning horizon.

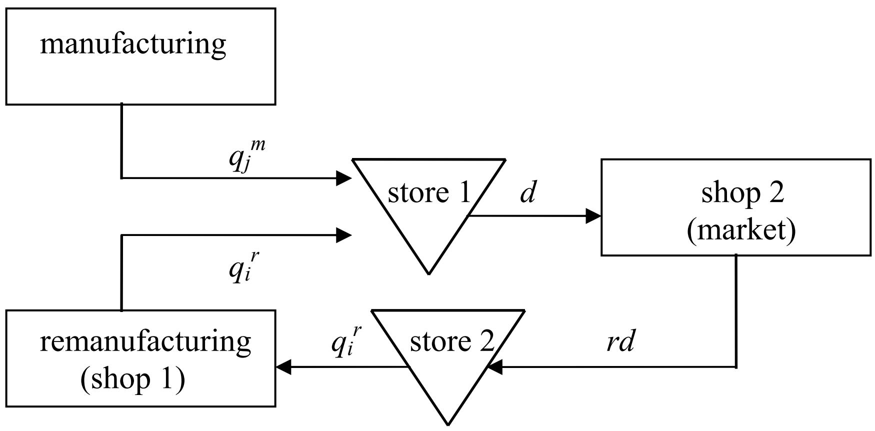
2. The model

We will investigate a two-store reverse logistics model with continuous disposal. The demand is satisfied from store 1. (See Figure 1.) The product can be manufactured or remanufactured. From the market returned items are stored in store 2, than they are remanufactured and stored as new product in store 1.

The decision maker will minimize the setup and linear inventory holding costs. There is no lead time in the model. The following parameters are in our model:

- T length of the planning horizon,
- d the rate of demand,
- r the return rate in the second store ($0 \le r \le 1$),
- h the inventory holding costs in the first store,
- u the inventory holding costs in the second store, $h > u$,
- K_m setup cost for a manufacturing batch,
- K_r setup cost for a remanufacturing batch.

Figure 1. The material flow in the model



It is assumed that the inventory holding costs for the end product are higher then the holding costs of the returned items $h > u$. The decision variables are the following:

- n number of manufacturing batches,
- m number of remanufacturing batches,
- I^1_k the inventory level in the first shop before arrival the kth batch,
- I^2_k the inventory level in the second shop before arrival the kth batch,
- q^m_j the ith manufacturing batch size,
- t^m_j length of time between the jth and $(i+1)$th manufacturing batches,
- q^r_i the ith remanufacturing batch size,
- t^r_i length of time between the ith and $(i+1)$th remanufacturing batches,
- T^c the cycle time.

To determine optimal size and number of manufacturing and remanufacturing batches, we follow the next way. First, we take a general possible inventory holding cost function $GIHC^c\left(\{I_i^1\}_{i=1}^{m+n}, I_m^2, \{t_i^r\}_{i=1}^m, \{t_j^m\}_{j=1}^n, m, n, T\right)$ under the assumption that the cycle time (length of a cycle) is equal to the length of the planning horizon and the numbers of batches are fixed. Then using the linearity of this general inventory holding cost function in the inventory levels, a new cost function $GHC^c\left(\{t_i^r\}_{i=1}^m, \{t_j^m\}_{j=1}^n, m, n, T\right)$ can be obtained with eliminating the inventory levels from this function. After this selection we determine the minimal inventory holding costs $HC^c(m,n,T)$ in a cycle. The constructed cost function depends on the numbers of batches and the cycle time, they are the parameters of the cost function. In solving the cost minimization problem, we use the results of the non-linear programming and the Lagrange-formulation. The sum of the setup costs and the minimal inventory is the total costs of a cycle $TC^c(m,n,T)$. Refinering the division of the planning horizon, the total cost function $TC(m,n,T^c)$ is detemined. With this construction we will show that the total cost approach leads to the average cost method in a known planning horizon. After determining the optimal cycle time in dependence on the numbers of manufacturing and remanufacturing batches, the cost function $C(m,n)$ is analyzed. The optimal numbers of the batches are calculated with the help of an auxiliary problem.

2.1. Construction of the inventory holding costs in a cycle

First, let us assume that the number of manufaturing and remanufacturing batches, say n and m, are fixed. It is assumed that the first cycle time is equal to the planning horizon.

The inventory level in the first shop consists of two terms: the levels during remanufacturing and a manufacturing batches. A possible case shown in Figure 2. In order to determine the inventory level at any time during the planning horizon, we must unite the sets of batch sizes and time of point of remanufacturing and manufacturing activities, and we make no difference whether a remanufacturing or manufacturing batch is arrived in the store 1.

The inventory holding costs can be calculated as the integral of the inventory levels. Let us assume that the inventory level function is function $f(t)$ in store 1 and function $g(t)$ in store 2. This functions are piecewise linear. The level function are defined as follows:

$$f(t)=\begin{cases} I_{i-1}^1+q_i^r-dt & t\in\left[\sum_{k=1}^{i-1}t_k^r,\sum_{k=1}^{i}t_k^r\right),\ i=1,\dots,m \\ I_{m+j-1}^1+q_j^m-dt & t\in\left[\sum_{k=1}^{j-1}t_k^m,\sum_{k=1}^{j}t_k^m\right),\ j=1,\dots,n \end{cases},$$

and

$$g(t)=\begin{cases} I_{i-1}^2-q_i^r+rdt & t\in\left[\sum_{k=1}^{i-1}t_k^r,\sum_{k=1}^{i}t_k^r\right),i=1,\dots,m \\ I_m^2+rd\left(t-\sum_{i=1}^{m}t_i^r\right) & t\in\left[\sum_{i=1}^{m}t_i^r,T\right] \end{cases}.$$

The holding costs are than

$$h\int_0^T f(t)dt+u\int_0^T g(t)dt\,.$$

Let us introduce a new parameter T_1, where $T_1 = \sum_{i=1}^{m-1} t_i^r$. It is remanufactured before point of time T_1, and it will be manufactured after that point. With this assumption the integral can be written as

$$h\int_0^T f(t)dt + u\int_0^T g(t)dt = (h-u)\int_0^{T_1} f(t)dt + h\int_{T_1}^T f(t)dt + u\int_0^{T_1}\left[f(t)+g(t)\right]dt + u\int_{T_1}^T g(t)dt$$

Let us note that the function $f(t)+g(t)$ is a linear funtion on $[0,T_1]$. This fact makes easier to calculate the holding costs.

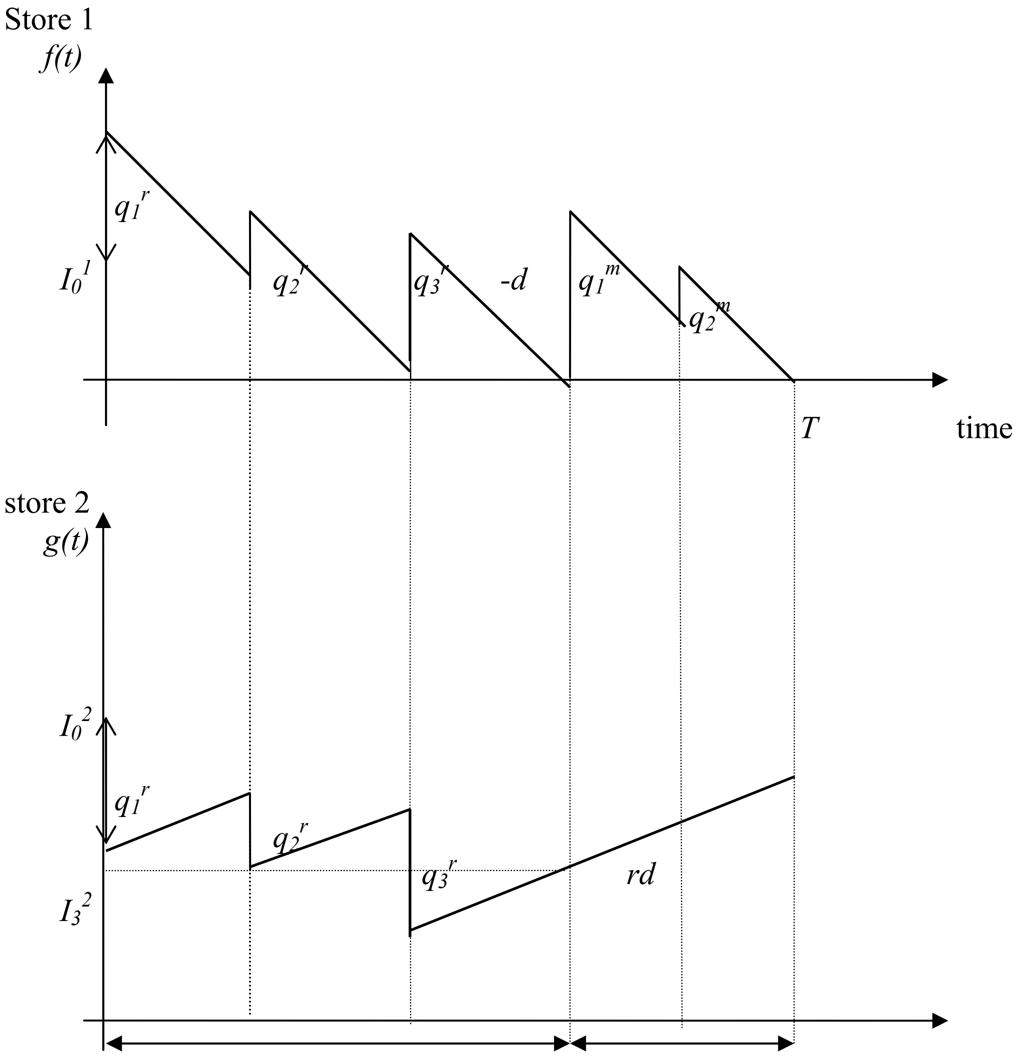
The inventory holding costs for store 1 is defined as follows. Let us introduce the initial inventory level I^1_k at the beginning of period k. Then the stock-flow identity can be written

$$I_i^1 = I_{i-1}^1 + q_i^r - dt_i^r, \qquad (i=1,..,m),$$

and

$$I_{m+j}^1 = I_{m+j-1}^1 + q_j^m - dt_j^m, \qquad (j=1,..,n),$$

Figure 2. The inventory level functions for a possible cycle with *m=3* and *n=2*



t_1^r t_2^r t_3^r t_1^m t_2^m

T time

Remanufacturing Manufacturing

and it is assumed that the first arrival in store 1 occures at time points 0, and the initial inventory level is zero: $I_1^1 = 0$. Remanufacturing occures before point of time T_1, so the length of time manufacturing and remanufacturing are

$$T_1 = \sum_{i=1}^{m-1} t_i^r ,$$

and

$$T - T_1 = \sum_{j=1}^{n} t_j^m .$$

The average inventory level in period k for store 1 is the following

$$\bar{I}_k^1 = \frac{I_{k-1}^1 + q_k^l + I_k^1}{2} = \frac{\left(I_{k-1}^1 + q_k^l - dt_k^l\right) + I_k^1 + dt_k^l}{2} = I_k^1 + \frac{d}{2} t_k^l ,$$

where $l = r$, if $k \le m$ and $l = m$, if $k > m$.

The total holding cost for store 1

$$h\sum_{k=1}^{m+n} \bar{I}_k^1 t_k = h\sum_{i=1}^{m}\left(I_i^1 + \frac{d}{2}t_i^r\right)t_i^r + h\sum_{j=1}^{n}\left(I_{m+j}^1 + \frac{d}{2}t_j^m\right)t_j^m = h\int_0^{T_1} f(t)dt + h\int_{T_1}^{T} f(t)dt .$$

Let us now examine the holding costs for store 2. The stock-flow identity is in this case for period k

$$I_i^2 = I_{i-1}^2 - q_i^r + rdt_i^r , \qquad (i=1,..,m).$$

The inventory level is a linear increasing function after point of time T_1. If we notice that the sum of the inventory levels is a linear decreasing function, then

$$u\int_0^{T_1}\left[f(t) + g(t)\right]dt = u\left[\left(I_m^1 + I_m^2\right) + (1-r)\frac{d}{2}T_1\right]T_1 ,$$

and

$$u\int_{T_1}^{T} g(t)t = u\left[I_m^2 + r\frac{d}{2}\left(T - T_1\right)\right]\left(T - T_1\right)$$

The total inventory holding costs for this reverse logistics are

$$GIHC^{c}\left(\left\{I_i^1\right\}_{i=1}^{m+n}, I_{m-1}^2, q_m^r, \left\{t_i^r\right\}_{i=1}^{m}, \left\{t_j^m\right\}_{j=1}^{n}, T_1, m, n, T\right)=$$
$$(h-u)\sum_{i=1}^{m}\left(I_i^1+\frac{d}{2}t_i^r\right)t_i^r + h\sum_{j=1}^{n}\left(I_{m+j}^1+\frac{d}{2}t_j^m\right)t_j^m + \qquad .$$
$$+u\left[\left(I_m^1+I_m^2\right)+(1-r)\frac{d}{2}T_1\right]T_1 + u\left[I_m^2+r\frac{d}{2}\left(T-T_1\right)\right]\left(T-T_1\right)$$

We must minimize the total inventory holding costs under the conditions:

$$T_1=\sum_{i=1}^{m}t_i^r ,$$

$$T-T_1=\sum_{j=1}^{n}t_j^m ,$$

and

$$I_k^1 \geq 0 \qquad (k=1,...m+n),$$

$$t_i^r \geq 0 \qquad (i=1,...,m),$$

$$t_j^m \geq 0 \qquad (j=1,...,n)$$

$$I_m^2 \geq rdt_m^r .$$

2.2. The optimal inventory holding policy in a cycle

Before solving the problem, let us supply some characteristics of the optimal inventory strategy.

The optimal inventory holding policy can be characterized by

Proposition 1.:

In the optimal strategy:

$$\left(I_k^1\right)^0=0,\ k=1,...,\ m+n$$

and

$$\left(I_m^2\right)^0=rdt_m^r .$$

The proof is obvious. The holding costs are linear in the inventory levels, so they must take their minimal values in the optimal solution. It follows from the proposition immediately, that the remanufacturing and manufacturing lots are

$$q_i^r = dt_i^r \qquad (i=1,...,m),$$
$$q_j^m = dt_j^m \qquad (j=1,...n).$$

A second result gives information about the length of time of remanufacturing and manufacturing.

Proposition 2.:

In the optimal solution: $T_1 = rT$ and $T - T_1 = (1-r)T$.

Proof. The initial inventory level is zero in the first shop, and the initial and ending inventory levels are equal in the second shop. The ending inventory level in the second shop is equal to value $g(T)$, i.e.

$$g(T) = \left(I_m^2\right)^0 + rd(T - T_1)\,.$$

The value $f(0)+g(0)$ is equal to the initial inventory level in the second shop, because the initial inventory level is zero in the first shop. The initial level for the second shop can be written in the following

$$g(0) = \left(I_m^1\right)^0 + \left(I_m^2\right)^0 + (1-r)T_1\,,$$

but the optimal inventory level before an ordering is equal to zero, and this fact proves the proposition.

The meaning of the proposition is that the time requirement of the remanufacturing is equal to the multiplication of the return rate and the length of the planning horizon. With the help of the propositions, the general inventory holding cost function can be simplified in the following way:

$$GHC^c\left(\left\{t_i^r\right\}_{i=1}^m, \left\{t_j^m\right\}_{j=1}^n, m, n, T\right) =$$
$$(h-u)\frac{d}{2}\sum_{i=1}^m \left(t_i^r\right)^2 + h\frac{d}{2}\sum_{j=1}^n \left(t_j^m\right)^2 + urdT \cdot t_m^r + u\frac{d}{2}r(1-r)T^2\,.$$

In the next step the time intervals between two succeeding batches are minimized:

$$GHC^c\left(\left\{t_i^r\right\}_{i=1}^m, \left\{t_j^m\right\}_{j=1}^n, m, n, T\right) \to \min\,,$$

such that

$$rT = \sum_{i=1}^{m} t_i^r ,$$

$$(1-r)T = \sum_{j=1}^{n} t_j^m .$$

To solve the problem, we use the standard results of the non-linear mathematical programming with the condition that the last remanufacturing batch t_m^r is fixed. The Lagrangian of the problem is

$$L\left(\left\{t_i^r\right\}_{i=1}^{m-1}, \left\{t_j^m\right\}_{j=1}^{n}, \lambda_1, \lambda_2\right) = (h-u)\frac{d}{2}\sum_{i=1}^{m-1}\left(t_i^r\right)^2 + h\frac{d}{2}\sum_{j=1}^{n}\left(t_j^m\right)^2 + u\frac{d}{2}r(1-r)T^2$$
$$+(h-u)\frac{d}{2}\left(t_m^r\right)^2 + urdT\cdot t_m^r + \lambda_1\left((rT - t_m^r) - \sum_{i=1}^{m-1} t_i^r\right) + \lambda_2\left((1-r)T - \sum_{j=1}^{n} t_j^m\right).$$

After differentiating the Lagrangian, we have the following necessary conditions of optimality:

$$\frac{\partial L}{\partial t_i^r} = (h-u)dt_i^r - \lambda_1 = 0 \qquad (i = 1,\dots,m-1),$$

$$\frac{\partial L}{\partial t_j^m} = hdt_j^m - \lambda_2 = 0 \qquad (j = 1,\dots,n),$$

The time intervals between optimal manufacturing and remanufacturing batches are easy to calculate. The intervals are equal to the remanufacturing and manufacturing activities:

$$t_r^r = t^r = \frac{rT - t_m^r}{m-1} \qquad (i = 1,\dots,m-1),$$

$$t_j^m = t^m = \frac{1-r}{n}T \qquad (j = 1,\dots,n).$$

Let us now calculate the inventory holding costs in dependence on the last remanufacturing batch. The cost function has the following form after the substitution of the optimal time intervals:

$$GHC^c\left(t_m^r, m, n, T\right) = (h-u)\frac{d}{2}\frac{m}{m-1}\left(t_m^r\right)^2 + rdT\left(u - \frac{h-u}{m-1}\right)\cdot t_m^r +$$
$$\left[(h-u)\frac{r^2}{m-1} + h\frac{(1-r)^2}{n} + ur(1-r)\right]\frac{d}{2}T^2 .$$

It can be given a lower bound for the remanufacturing batch t_m^r. The maximal (ending) inventory level in the second store is equal to the following expression

$$I_{\max}^{2} = rd\left(t_{m}^{r} + \sum_{j=1}^{n} t_{j}^{m} \right) = rd\left(t_{m}^{r} + (1-r)T\right).$$

Because of the equality of the remanufacturing batches, the last but one remanufacturing batch takes its minimal value, but this minimal value must be non-negative.

$$I_{\max}^{2} - (m-1)d\,\frac{rT - t_{m}^{r}}{m-1} + (m-2)rd\,\frac{rT - t_{m}^{r}}{m-1} \geq 0 .$$

These two expression gives us the lower bound for the length of the last remanufacturing batch:

$$t_{m}^{r} \geq \frac{r}{m-1+r} rT .$$

For fixed number of remanufacturing batches m, we must minimize a quadratic function on a bounded interval. We examine two cases.

Case 1.: $$\frac{h-um}{(h-u)m} rT > \frac{r}{m-1+r} rT .$$

In this case the quadratic function takes its minimum inside of the interval:

$$\left(t_{m}^{r}\right)^{0} = \frac{h-um}{(h-u)m} rT .$$

The cost function can be written after substitution as

$$GHC^{c}\left(\left(t_{m}^{r}\right)^{0}, m, n, T\right) = \left[\frac{h^{2}}{h-u} r^{2} \frac{1}{m} + h(1-r)^{2} \frac{1}{n} + ur\,\frac{h(1-r)-u}{h-u} \right] \frac{d}{2} T^{2} .$$

This inventory holding cost function is defined for m, if

$$1 \leq m < \frac{h}{u}(1-r) .$$

If the expression $\frac{h}{u}(1-r)$ is smaller than one, then this case can not occur and it must be examined the next case.

Case 2.: $$\frac{h-um}{(h-u)m} rT \leq \frac{r}{m-1+r} rT .$$

In this case the quadratic function takes its minimum on its lower bound:

$$\left(t_m^r\right)^0 = \frac{r}{m-1+r} rT .$$

The cost function is after substitution:

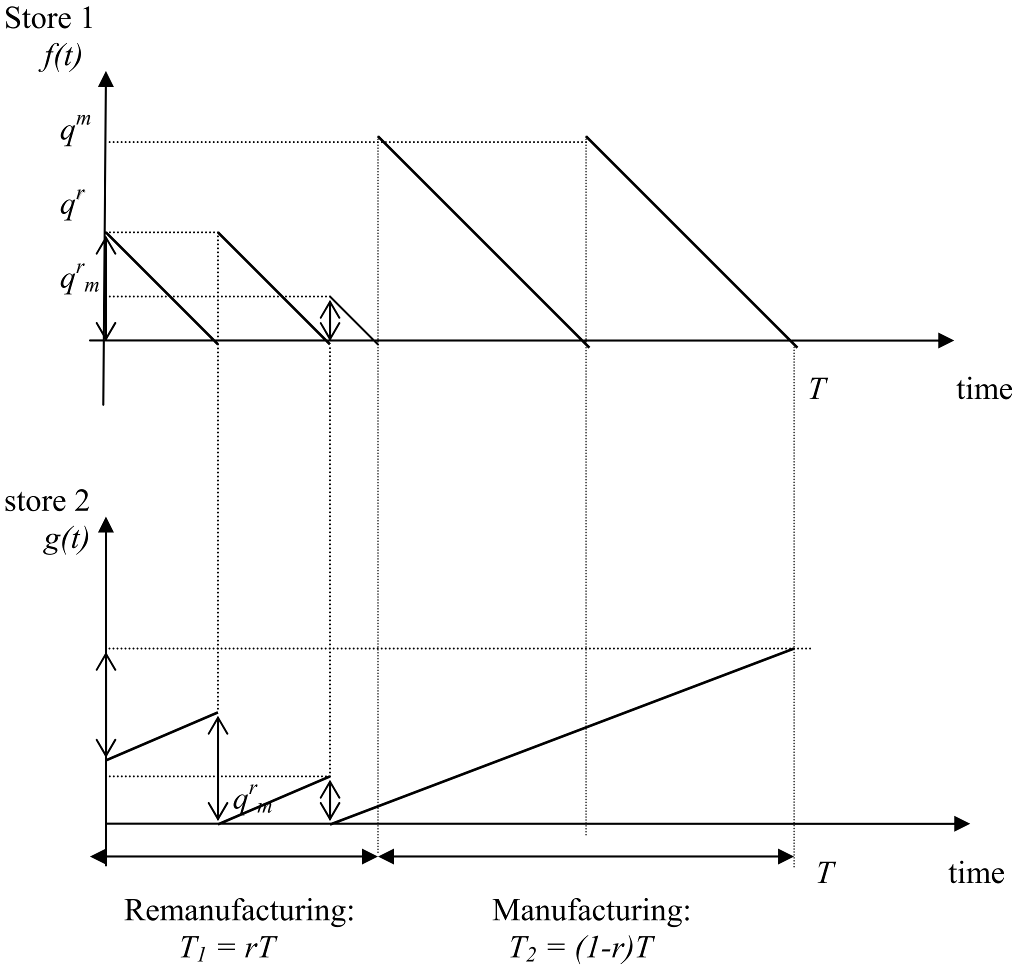
$$GHC^c\left(\left(t_m^r\right)^0, m, n, T\right) =$$
$$\left\{\left[\frac{h-u+2ur}{m-1+r} - \frac{(h-u)r(1-r)}{(m-1+r)^2}\right] r^2 + h\frac{(1-r)^2}{n} + ur(1-r)\right\}\frac{d}{2}T^2 .$$

This new holding cost function is defined for the following values of m

$$m \geq \max\left\{\frac{h}{u}(1-r), 1\right\}.$$

It is true the following

Figure 3. The optimal inventory holding strategy for a cycle with *m=3* and *n=2* (*h > u*)



Theorem 1.:

The optimal inventory policy for a reverse logistics system is to order, if the inventory level is zero in the first shop. In a cycle with given number of setups for remanufacturing m and manufacturing n, the optimal lot sizes are *(m-1)* equal remanufacturing q^r and n manufacturing batches q^m. The last remanufacturing batch q_m^r before the manufacturing is less then *(m-1)* equal remanufacturing batches The optimal batch sizes are

$$q^m = dt^m = \frac{1-r}{n} dT ,$$

$$q^r = dt^r = \begin{cases} \dfrac{h}{(h-u)m} rdT & m < \dfrac{h}{u}(1-r) \\ \dfrac{r}{m-1+r} dT & m \geq \dfrac{h}{u}(1-r) \end{cases} .$$

$$q_m^r = dt_m^r = \begin{cases} \dfrac{h-um}{(h-u)m} rdT & m < \dfrac{h}{u}(1-r) \\ \dfrac{r^2}{m-1+r} dT & m \geq \dfrac{h}{u}(1-r) \end{cases}$$

The minimal inventory holding costs are in this cycle

$$HC^c(m,n,T) = \left[F(m) + h(1-r)^2 \frac{1}{n} \right] \frac{d}{2} T^2 ,$$

where

$$F(m) = \begin{cases} \dfrac{h^2}{h-u} r^2 \dfrac{1}{m} + ur \dfrac{h(1-r)-u}{h-u} & 0 < m < \dfrac{h}{u}(1-r) \\ \dfrac{h-u+2ur}{m-1+r} r^2 - \dfrac{(h-u)(1-r)r^3}{(m-1+r)^2} + ur(1-r) & m \geq \dfrac{h}{u}(1-r) \end{cases}$$

The total costs can be written as follows

$$TC^c(m,n,T) = (K_m n + K_r m) + \left[F(m) + h(1-r)^2 \frac{1}{n} \right] \frac{d}{2} T^2 .$$

2.3. The optimal cycle time and batch numbers

Now we investigate the dependence of the total inventory costs on the number of cycles. Let us assume that we divide the planning horizon into equidistant subintervals. After the jth step we have the following costs

$$TC\left(m,n,\frac{T}{j} \right) = j \cdot TC^c\left(m,n,\frac{T}{j} \right) = j\left\{ (K_m n + K_r m) + \left[F(m) + h(1-r)^2 \frac{1}{n} \right] \frac{d}{2} \left(\frac{T}{j} \right)^2 \right\},$$

where the function $TC\left(m,n,\frac{T}{j}\right)$ denotes the total costs, if the number of cycles is equal to j.

Let us now introduce the variable of the cycle time

$$T^c = \frac{T}{j}.$$

With this new variable the total cost function is

$$TC\left(m,n,T^c\right) = T\left\{\left(K_m n + K_r m\right)\frac{1}{T^c} + \left[F(m) + h(1-r)^2 \frac{1}{n}\right]\frac{d}{2}T^c\right\}$$

It is easy to calculate the average cost function $AC(m,n,T^c)$ after dividing by the length of the planning horizon. With this expression we wanted to demonstrate that the total cost and average cost approaches lead to the same result. We must not distinguish between these two methods.

$$AC(m,n,T^c) = \frac{TC(m,n,T^c)}{T} = \left(K_m n + K_r m\right)\frac{1}{T^c} + \left[F(m) + h(1-r)^2 \frac{1}{n}\right]\frac{d}{2}T^c$$

The new cost function is convex in the cycle time T^c, so the optimality is guaranteed. The optimal cycle time is

$$\left(T^c\right)^0 = \sqrt{\frac{2}{d}}\sqrt{\frac{K_m n + K_r m}{F(m) + h(1-r)^2 \frac{1}{n}}}.$$

The last total cost function can be written in the following form

$$C(m,n) = T\sqrt{2d}\sqrt{S(m,n)}$$

where

$$S(m,n) = \left(K_m n + K_r m\right)\left[F(m) + h(1-r)^2 \frac{1}{n}\right].$$

With this reformulation a new auxiliary problem can be defined

$$S(m,n) \to \min$$

such that

$$m \geq 1,\ n \geq 1.$$

This problem is an integer mathematical programming model. In this paper we investigate the continuous version of the problem, we assume that the variables m and n are continuous. First we examine those cases when the optimal solutions are inside of the possible set, i.e. $m > 1$ and $n > 1$. In this case the partial derivatives are equal to zero:

$$\frac{d}{dm}S(m,n) = K_r h(1-r)^2 \frac{1}{n} + K_m F'(m)\cdot n + K_r[m\cdot F(m)]' = 0$$
$$\frac{d}{dn}S(m,n) = -K_r h(1-r)^2 m\cdot\frac{1}{n^2} + K_m F(m) = 0 .$$

After simple manipulation we have necessary and sufficient conditions of optimality in the next form

$$[m\cdot F(m)]' = 0$$
$$n(m) = (1-r)\sqrt{\frac{K_r h}{K_m}\frac{m}{F(m)}} .$$

It is easy to show that the function $m\cdot F(m)$ is convex and there exists an optimal value m. So it is true

Proposition 3.:

If the next inequalities

$$F'(1) + F(1) \le 0$$

and

$$\frac{K_r h}{K_m}(1-r)^2 \ge F(1)$$

hold then the optimal solution is inside of the possible set.

The proof is obvious. The first inequality shows that the function $m\cdot F(m)$ is monotonously decreasing in point one, the second inequality is the condition for n.

Example. Let $K_m = 1$, $K_r = 2.2\cdot 10^8$, $h = 200$, $u = 20$ and $r = 0.9999$. Then the optimal solution is $(m^o,n^o) = (1.414, 2)$ and $S(1.414, 2) = 483.907\cdot 10^8$. This example shows that the optimal solution can be for extreme cost parameters inside of the possible set of (m,n).

Let us now assume that the conditions of Proposition 3. do not hold. In this case the optimal solutions ly on the on the lines either $m = 1$ or $n = 1$. We introduce the next parameter

$$m_0 = \{m \mid K_m F(m) - K_r h(1-r)^2 m = 0;\ m > 0\}.$$

The parameter m_0 is the point for which batch number n is equal to one. This value can be smaller than one.

The optimal solution charaterizes the

Theorem 2.:

The optimal number of batches

(i) $m^o = 1, n^o = (1-r)\sqrt{\frac{K_r h}{K_m}\frac{1}{F(1)}}$, if $m_0 < 1$ and $S'_m(1, n(1)) > 0$,

(ii) $m^o = 1, n^o = 1$, if $m_0 \geq 1$ and $S'_m(1, n(1)) > 0$,

(iii) $m^o = m^*, n^o = 1$, if $m_0 \geq 1$, $S'_m(1, n(1)) \leq 0$ and $S'_m(m^*, n(m^*)) = 0$.

Proof. To prove the theorem we write the function *S(m,n)* in the following form

$$S(m,n) = K_r h(1-r)^2 m \cdot \frac{1}{n} + K_m F(m) \cdot n + K_r F(m) \cdot m + K_m h(1-r)^2 .$$

First we determine the optimal value *n*. This function is strictly convex in *n*, so the optimal *n* is

$$n(m) = \begin{cases} 1 & m < m_0 \\ (1-r)\sqrt{\frac{K_r h}{K_m}\frac{m}{F(m)}} & m \geq m_0 \end{cases}$$

and after substitution

$$S(m, n(m)) = \begin{cases} K_r h(1-r)^2 m + K_m F(m) + K_r F(m) \cdot m + K_m h(1-r)^2 & m < m_0 \\ \left(\sqrt{K_r F(m) \cdot m} + (1-r)\sqrt{K_m h}\right)^2 & m \geq m_0 \end{cases}.$$

The function $S(m, n(m))$ is differentiable for all positive *m* and the derivative $S'_m(m, n(m))$ is continuous function. If value m_0 smaller than one then the manufacturing batch number *n* greater than one, and condition (i) is proved. The other conditions can be proved in the similar way.

The continuous problem is totally solved with these results. In the next section we demonstrate the results of our model.

3. Numerical examples

Let us first compare a predetermined inventory holding policy with the offered optimal policy. The predetermined policy was supplied in paper [13]. In this case the last remanufacturing batch size is equal to the last batch.

Example 1.: In paper [1] the following example was investigated in the terms used here:

$K_m = \$ 750$, $K_r = \$ 100$, $h = \$ 200$, $u = \$ 20$, $d = 1{,}000$ units per year.

The optimal parameters are presented below in Table 1.

The optimal number of batches are the same for the two policies. If the optimal batch number of the remanufacturing is equal to one, then there is no difference between the two strategies. Comparing the cost savings we can say thet the maximum of saving is not greater than one percent for this parameters.

Table 1. Comparison of the predetermined and optimal policies in Example 1.

Return rate r	**Total costs** Optimal policy	**Total costs** Predetermined policy	**Batch numbers** (m^o, n^o)	**Cost saving** ΔS (%)
0.15	15,975	15,975	(1,2)	0.0
0.2	15,427.2	15,427.2	(1,1)	0.0
0.4	13,379.9	13,392.5	(2,1)	0.094
0.6	11,349	11,409.6	(4,1)	0.534
0.8	9,326.6	9,406.38	(10,1)	0.855
0.9	8,307.12	8,357.54	(19,1)	0.607
0.95	7,748.72	7,776.38	(32,1)	0.357

Example 2.: Let us now investigate the following parapeters:

$K_m = \$ 100$, $K_r = \$ 750$, $h = \$ 200$, $u = \$ 20$, $d = 1{,}000$ units per year.

The optimal parameters are presented below in Table 2.

Table 2. Comparison of the predetermined and optimal policies in Example 2.

Return rate r	**Total costs** Optimal policy	**Total costs** Predetermined policy	**Batch numbers** (m^o, n^o)	**Cost saving** ΔS (%)
0.15	8,730	8,730	(1,12)	0.0
0.2	9,302.3	9,302.3	(1,9)	0.0
0.4	11,549.9	11,549.9	(1,4)	0.0
0.6	13,784	13,784	(1,2)	0.0
0.8	16,074.8	16,074.8	(1,1)	0.0
0.9	17,148.8	17,234.8	(3,1)	0.502
0.95	17,740.9	17,797.3	(4,1)	0.318

For this example we can state the same as before, the maximal saving is a half percent and batch numbers are the same.

4. Conclusions and further research

In this paper the optimal (re)manufacturing-inventory strategy was supplied to model [1], in case of $h > u$. This strategy is called „substitution policy“. It is easy to construct in a similar way the optimal policy for the case of $h < u$. (See Appendix of this paper.) This policy is named „continuous supplement policy“. The offered optimal policy differs from the model [1] only in a „switching“ batch size between remanufacturing and manufacturing. The optimal numbers of remanufacturing and manufacturing batches can be determined with a meta-model based on the results offered in [14].

The numerical results show that the predetermined policy, as sub-optimal strategy, gives a good estimation of the optimal inventory holding policy. In the investigated examples the cost savings were smaller than one percent. It comes into question whether it is profitable to apply a sophisticated method to determine the optimal inventory holding policy.

The results of the paper can be generalized in two directions. A first generalization could be the examination of the integer solution. We have given only the continuous solution of the meta-model. The dependence of the integer solution on the cost parameters was here not investigated. A second direction is the analyzis of the optimal solution in dependence of the remanufacturing and manufacturing batch sizes and numbers on the return rate, if it is a decision variable, as it was made in [4].

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Appendix:

Optimal policy for the „continuous supplement policy"

1. Construction of the cost function

Let us apply the nototion of the paper to this strategy. If $u > h$

$$u\int_0^T g(t)dt + h\int_0^T f(t)dt = (u-h)\int_0^{T_1} g(t)dt + h\int_{T_1}^T f(t)dt + h\int_0^{T_1}\left[f(t)+g(t)\right]dt + u\int_{T_1}^T g(t)dt$$

$$(u-h)\int_0^{T_1} g(t)dt = (u-h)\sum_{i=1}^{m}\left(I_{i-1}^2 - q_i^r + r\frac{d}{2}t_i^r\right)t_i^r \ ,$$

$$h\int_{T_1}^T f(t)dt = h\sum_{j=1}^{n}\left(I_{m+j}^1 + \frac{d}{2}t_j^m\right)t_j^m \ ,$$

$$h\int_0^{T_1}\left[f(t)+g(t)\right]dt = h\left[\left(I_m^1 + I_m^2\right) + (1-r)\frac{d}{2}T_1\right]T_1$$

$$u\int_{T_1}^T g(t)dt = u\left[I_m^2 + r\frac{d}{2}\left(T-T_1\right)\right]\left(T-T_1\right)$$

The total inventory holding costs for this reverse logistics are

$$(u-h)\sum_{i=1}^{m}\left(I_{i-1}^2 - q_i^r + r\frac{d}{2}t_i^r\right)t_i^r + h\sum_{j=1}^{n}\left(I_{m+j}^1 + \frac{d}{2}t_j^m\right)t_j^m +$$
$$+ h\left[\left(I_m^1 + I_m^2\right) + (1-r)\frac{d}{2}T_1\right]T_1 + u\left[I_m^2 + r\frac{d}{2}\left(T-T_1\right)\right]\left(T-T_1\right)$$

We must minimize the total inventory holding costs under the conditions:

$$T_1 = \sum_{i=1}^{m} t_i^r \ ,$$

$$T - T_1 = \sum_{j=1}^{n} t_j^m \ ,$$

and

$I_{i-1}^2 \geq q_i^r$ *(i=1,...,m)*,

$I_m^2 \geq rdt_m^r$,

$I_{m+j}^1 \geq 0$ *(j=0,1,..,n)*,

$t_i^r \geq 0$ *(i=1,...,m)*,

$t_j^m \geq 0$ *(j=1,...,n)*.

2. The optimal inventory holding policy in a cycle

Before solving the problem, let us supply some characteristics of the optimal solution.

The optimal inventory holding policy can be characterized by

Proposition 1.:

In the optimal strategy:

$$\left(I_k^1\right)^0 = 0, \qquad k = m,..., m+n,$$

$$\left(I_{i-1}^2\right)^0 = q_i^r \qquad i = 1, ...m,$$

and

$$\left(I_m^2\right)^0 = rdt_m^r.$$

The proof is obvious. The holding costs are linear in the inventory levels, so they take their minimal values in the optimal solution. It follows from the proposition immediately, that the remanufacturing and manufacturing lots are

$$q_1^r = rd\left[t_m^r + (T - T_1)\right],$$
$$q_i^r = rdt_{i-1}^r \qquad (i=2,...,m),$$
$$q_j^m = dt_j^m \qquad (j=1,...n).$$

A second result is

Proposition 2.:

In the optimal solution: $T_1 = rT$ and $T - T_1 = (1 - r)T$.

Proof. The initial and ending inventory levels are equal in the first and second shops. The ending inventory level in the second shop is equal to value *g(T)*, i.e.

$$g(T) = \left(I_m^2\right)^0 + rd(T - T_1)\ .$$

The value *f(0)+g(0)* is equal to the initial inventory level in the second shop, because the initial inventory level is zero in the first shop. The initial level for the second shop can be written in the following

$$g(0) = \left(I_m^1\right)^0 + \left(I_m^2\right)^0 + (1-r)dT_1\,,$$

but the optimal inventory level before an ordering is equal to zero, and this fact proves the proposition.

With the help of the propositions, the problem can be simplified in the following way:

$$\begin{aligned}&(u-h)r\frac{d}{2}\sum_{i=1}^{m}\left(t_i^r\right)^2 + \left[hr + u(1-r)\right]rdT\cdot t_m^r + h\frac{d}{2}\sum_{j=1}^{n}\left(t_j^m\right)^2 + \\ &+\left[hr+u(1-r)\right]r(1-r)\frac{d}{2}T^2 \qquad \to \min\end{aligned},$$

such that

$$rT - t_m^r = \sum_{i=1}^{m-1} t_i^r\ ,$$

$$(1-r)T = \sum_{j=1}^{n} t_j^m\ .$$

To solve the problem, we use the standard results of the non-linear mathematical programming. The Lagrangian of the problem is

$$\begin{aligned}&L\left(\left\{t_i^r\right\}_{i=1}^{m-1}, \left\{t_j^m\right\}_{j=1}^{n}, \lambda_1, \lambda_2\right) = (u-h)r\frac{d}{2}\sum_{i=1}^{m}\left(t_i^r\right)^2 + \left[hr+u(1-r)\right]rdT\cdot t_m^r + h\frac{d}{2}\sum_{j=1}^{n}\left(t_j^m\right)^2 + \\ &+\left[hr+u(1-r)\right]r(1-r)\frac{d}{2}T^2 + \lambda_1\left(\left(rT - t_m^r\right) - \sum_{k=1}^{m} t_k\right) + \lambda_2\left((1-r)T - \sum_{k=m+1}^{m+n} t_k\right)\end{aligned}.$$

After differentiating the Lagrangian, we have the following necessary conditions of optimality:

$$\frac{\partial L}{\partial t_i^r} = (u-h)rdt_i^r - \lambda_1 = 0 \quad (i = 1,\ldots,m\text{-}1),$$

$$\frac{\partial L}{\partial t_j^m} = hdt_j^m - \lambda_2 = 0 \qquad (j = 1,...,n).$$

The optimal remanufacturing and manufacturing batches are easy to calculate:

$$t_i^r = t^r = \frac{rT - t_m^r}{m-1} \qquad (i = 1,...,m-1).$$

$$t_j^m = t^m = (1-r)\frac{T}{n} \qquad (j = 1,...,n).$$

Let us now calculate the inventory holding costs in dependence on the last remanufacturing batch. The cost function has the following form after the substitution of the optimal time intervals:

$$GHC^c\left(t_m^r, m, n, T\right) = (u-h)r\frac{d}{2}\frac{m}{m-1}\left(t_m^r\right)^2 + \left(\left[hr + u(1-r)\right] - \frac{(u-h)r}{m-1}\right)rdT \cdot t_m^r + \left[(u-h)\frac{r^3}{m-1} + h\frac{(1-r)^2}{n} + \left[hr + u(1-r)\right]r(1-r)\right]\frac{d}{2}T^2 .$$

It can be given a lower bound for the remanufacturing batch t_m^r. The maximal (ending) inventory level in the second store is equal to the initial batch size in the first store:

$$I_{\max}^2 = rd\left(t_m^r + \sum_{j=1}^n t_j^m\right) = rd\left(t_m^r + (1-r)T\right) = q_1^r .$$

Because of the equality of the manufacturing batches, the last but one manufacturing batch takes its minimal value, but this minimal value must be non-negative.

$$rd\left(t_m^r + (1-r)T\right) + (m-2)rd\frac{rT - t_m^r}{m-1} - (m-1)d\frac{rT - t_m^r}{m-1} \geq 0 .$$

These two expression gives us the lower bound for the length of the last remanufacturing batch:

$$t_m^r \geq \frac{r}{m-1+r}rT .$$

For fixed number of remanufacturing batches m, we must minimize a quadratic function on a bounded interval. We examine two cases.

Case 1.: $$\frac{u - \left[hr + u(1-r)\right]m}{(u-h)m}T > \frac{r}{m-1+r}rT .$$

This case can not occur, because the difference is always non-positive for every $m \geq 1$. It is easy to check.

$$\frac{u-[hr+u(1-r)]m}{(u-h)m}T-\frac{r}{m-1+r}rT=-[hr+u(1-r)](m-1)\left(m-\frac{u(1-r)}{hr+u(1-r)}\right)T.$$

The value $\frac{u(1-r)}{hr+u(1-r)}$ is smaller than one, so this case can be excluded.

Case 2.: $\frac{u-[hr+u(1-r)]m}{(u-h)m}T \le \frac{r}{m-1+r}rT$.

In this case the quadratic function takes its minimum on its lower bound:

$$\left(t_m^r\right)^0=\frac{r}{m-1+r}rT.$$

The cost function is after substitution:

$$HC^c(m,n,T)=\left[F(m)+h(1-r)^2\frac{1}{n}\right]\frac{d}{2}T^2,$$

where

$$F(m)=-(u-h)r^2\left[\frac{r^2}{m-1+r}+(1-r)\right]^4+(u-h)r\left[\frac{r^2}{m-1+r}+(1-r)\right]^3+ \\ +(u-h)r\left[\frac{r^2}{m-1+r}+(1-r)\right]^2+2hr\left[\frac{r^2}{m-1+r}+(1-r)\right]+hr(1-r).$$

This new holding cost function is defined for all possible values of *m*.

It is true the following

Theorem 1.:

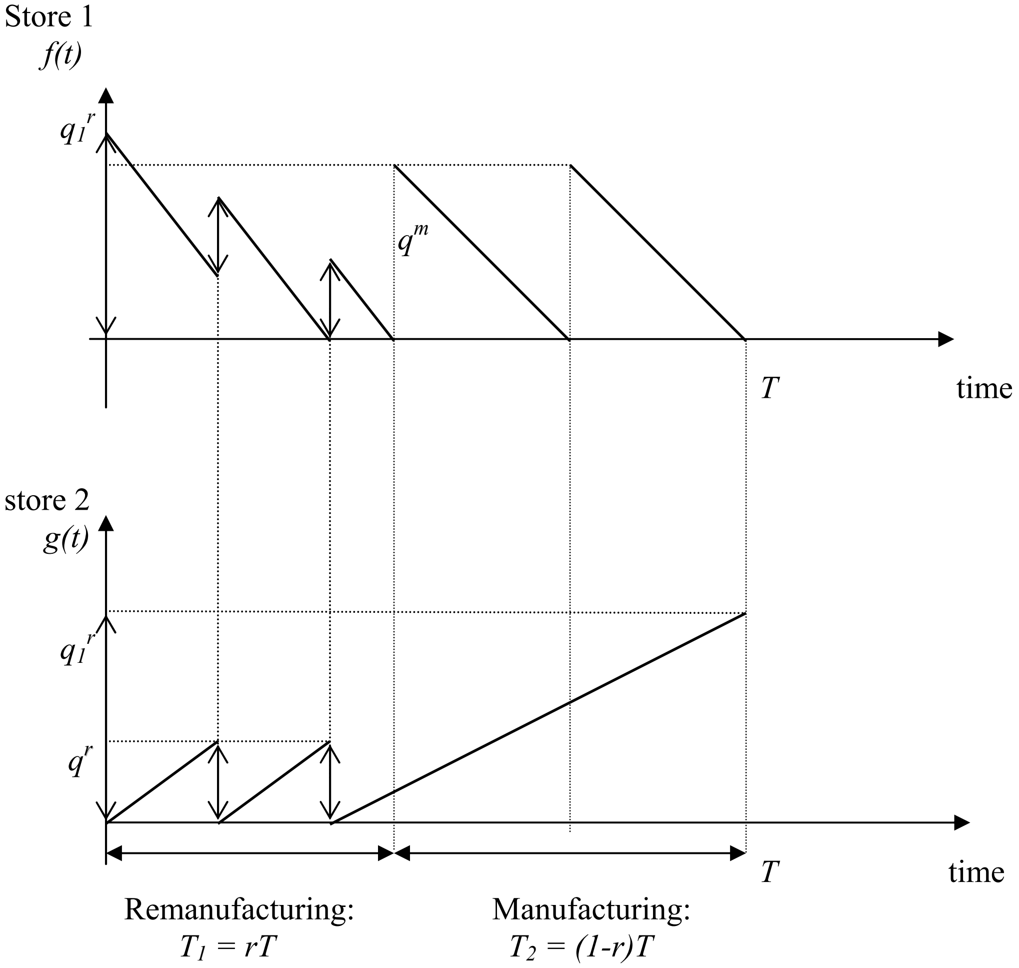
The optimal inventory policy for a reverse logistics system is to order, if the inventory level is zero in the first shop. In a cycle with given number of setups for remanufacturing *m* and manufacturing *n*, the optimal lot sizes are *(m-1)* equal remanufacturing q^r and *n* manufacturing batches q^m. The last remanufacturing batch $q_m{}^r$ before the manufacturing is less then *(m-1)* equal remanufacturing batches The optimal batch sizes are

$$q^m=dt^m=\frac{1-r}{n}dT,$$

$$q_1^r = rd\left(t_m^r + (1-r)T\right) = \left(\frac{r^2}{m-1+r} + (1-r)\right)rdT$$
.

$$q^r = rdt^r = \frac{r}{m-1+r}rdT$$

Figure 3. The optimal inventory holding strategy for a cycle with *m=3* and *n=2* (*u > h*)



The total costs can be written as follows

$$TC^c(m,n,T) = (K_m n + K_r m) + \left[F(m) + h(1-r)^2 \frac{1}{n} \right] \frac{d}{2} T^2 .$$

3. The optimal cycle time and batch numbers

Now we investigate the dependence of the total inventory costs on the number of cycles. Let us assume that we divide the planning horizon into equidistant subintervals. After the *j*th step we have the following costs

$$TC\left(m,n,\frac{T}{j} \right) = j \cdot TC^c\left(m,n,\frac{T}{j} \right) = j\left\{ (K_m n + K_r m) + \left[F(m) + h(1-r)^2 \frac{1}{n} \right] \frac{d}{2} \left(\frac{T}{j} \right)^2 \right\},$$

where the function $TC\left(m,n,\frac{T}{j} \right)$ denotes the total costs, if the number of cycles is equal to *j*.

Let us now introduce the variable of the cycle time

$$T^c = \frac{T}{j}.$$

With this new variable the total cost function is

$$TC\left(m,n,T^c\right) = T\left\{\left(K_m n + K_r m\right)\frac{1}{T^c} + \left[F(m) + h(1-r)^2 \frac{1}{n}\right]\frac{d}{2}T^c\right\}$$

It is easy to calculate the average cost function *AC(m,n,T^c^)* after dividing by the length of the planning horizon. With this expression we wanted to demonstrate that the total cost and average cost approaches lead to the same result. We must not distinguish between these two methods.

$$AC(m,n,T^c) = \frac{TC(m,n,T^c)}{T} = \left(K_m n + K_r m\right)\frac{1}{T^c} + \left[F(m) + h(1-r)^2 \frac{1}{n}\right]\frac{d}{2}T^c$$

The new cost function is convex in the cycle time T^c, so the optimality is guaranteed. The optimal cycle time is

$$\left(T^c\right)^0 = \sqrt{\frac{2}{d}}\sqrt{\frac{K_m n + K_r m}{F(m) + h(1-r)^2 \frac{1}{n}}}.$$

The last total cost function is

$$C(m,n) = T\sqrt{2d}\sqrt{\left(K_m n + K_r m\right)\cdot\left[F(m) + h(1-r)^2 \frac{1}{n}\right]}.$$

The last total cost function can be written in the following form

$$C(m,n) = T\sqrt{2d}\sqrt{S(m,n)}$$

where

$$S(m,n) = \left(K_m n + K_r m\right)\left[F(m) + h(1-r)^2 \frac{1}{n}\right].$$

With this reformulation a new auxiliary problem can be defined

$$S(m,n) \rightarrow \min$$

such that

$m \geq 1,\ n \geq 1$.

This problem is an integer mathematical programming model. In this paper we investigate the continuous version of the problem, we assume that the variables *m* and *n* are continuous. First we examine those cases when the optimal solutions are inside of the possible set, i.e. $m > 1$ and $n > 1$. In this case the partial derivatives are equal to zero:

$$\begin{aligned}
&\frac{d}{dm} S(m,n) = K_r h(1-r)^2 \frac{1}{n} + K_m F'(m) \cdot n + K_r [m \cdot F(m)]' = 0 \\
&\frac{d}{dn} S(m,n) = -K_r h(1-r)^2 m \cdot \frac{1}{n^2} + K_m F(m) = 0
\end{aligned}.$$

After simple manipulation we have necessary and sufficient conditions of optimality in the next form

$$\begin{aligned}
&[m \cdot F(m)]' = 0 \\
&n(m) = (1-r)\sqrt{\frac{K_r h}{K_m} \frac{m}{F(m)}}
\end{aligned}.$$

It is easy to show that the function $m \cdot F(m)$ is convex and there exists an optimal value *m*. So it is true

Proposition 3.:

If the next inequalities

$$F'(1) + F(1) \leq 0$$

and

$$\frac{K_r h}{K_m}(1-r)^2 \geq F(1)$$

hold then the optimal solution is inside of the possible set.

The proof is obvious. The first inequality shows that the function $m \cdot F(m)$ is monotonously decreasing in point one, the second inequality is the condition for *n*.

Let us now assume that the conditions of Proposition 3. do not hold. In this case the optimal solutions ly on the on the lines either $m = 1$ or $n = 1$. We introduce the next parameter

$$m_0 = \left\{ m \,\middle|\, K_m F(m) - K_r h(1-r)^2 m = 0;\ m > 0 \right\}.$$

The parameter m_0 is the point for which batch number *n* is equal to one. This value can be smaller than one.

The optimal solution charaterizes the

Theorem 2.:

The optimal number of batches

(i) $m^o = 1, n^o = (1-r)\sqrt{\frac{K_r h}{K_m}\frac{1}{F(1)}}$, if $m_0 < 1$ and $S'_m(1, n(1)) > 0$,

(ii) $m^o = 1, n^o = 1$, if $m_0 \geq 1$ and $S'_m(1, n(1)) > 0$,

(iii) $m^o = m^*, n^o = 1$, if $m_0 \geq 1$, $S'_m(1, n(1)) \leq 0$ and $S'_m(m^*, n(m^*)) = 0$.

Proof. To prove the theorem we write the function *S(m,n)* in the following form

$$S(m,n) = K_r h(1-r)^2 m \cdot \frac{1}{n} + K_m F(m) \cdot n + K_r F(m) \cdot m + K_m h(1-r)^2 .$$

First we determine the optimal value *n*. This function is strictly convex in *n*, so the optimal *n* is

$$n(m) = \begin{cases} 1 & m < m_0 \\ (1-r)\sqrt{\frac{K_r h}{K_m}\frac{m}{F(m)}} & m \geq m_0 \end{cases}$$

and after substitution

$$S(m, n(m)) = \begin{cases} K_r h(1-r)^2 m + K_m F(m) + K_r F(m) \cdot m + K_m h(1-r)^2 & m < m_0 \\ \left(\sqrt{K_r F(m) \cdot m} + (1-r)\sqrt{K_m h}\right)^2 & m \geq m_0 \end{cases} .$$

The function $S(m, n(m))$ is differentiable for all positive *m* and the derivative $S'_m(m, n(m))$ is continuous function. If value m_0 smaller than one then the manufacturing batch number *n* greater than one, and condition (i) is proved. The other conditions can be proved in the similar way. With this theorem we have supplied the optimal solution.