Műhelytanulmányok
Vállalatgazdaságtan Intézet

1093 Budapest, Fővám tér 8.
☎ (+36 1) 482-5566, Fax: 482-5567
www.uni-corvinus.hu/vallgazd

Supply chain coordination in a HMMS-type supply chain with purchasing

Imre Dobos

Barbara Gobsch

138. sz. Műhelytanulmány
HU ISSN 1786-3031

2011. április

Budapesti Corvinus Egyetem
Vállalatgazdaságtan Intézet
Fővám tér 8.
H-1093 Budapest
Hungary

Supply chain coordination in a HMMS-type supply chain with purchasing price contract

Imre Dobos[1]
Barbara Gobsch[2]

[1] Corvinus University of Budapest, Institute of Business Economics, H-1093 Budapest, Fővám tér 8., Hungary, imre.dobos@uni-corvinus.hu
[2] Europa-Universität Viadrina Frankfurt (Oder), Professur für Allgemeine Betriebswirtschaftslehre, insbesondere Industriebetriebslehre, D-15230 Frankfurt (Oder), Große Scharrnstr. 59., Germany, Gobsch@europa-uni.de

Absztrakt.
Egy ellátási láncot vizsgálunk egy beszállítóval és egy termelővel. A termelő kereslete időben ismert. A termelő és a beszállító költségfüggvénye kvadratikus termelési és készlettartási költségből áll. Ezen kívül értelmezünk egy lineáris beszerzési költséget termelő és a beszállító között. A modell ebben az értelmezésben egy differenciáljátékként értelmezhető. A beszállító döntési változója az értékesítési (beszerzési) ár és a termelési rátája, míg a termelőnek a beszerzési mennyiség és a termelési ráta. A problémát, mint egy Nash-játékot és mint kooperatív játékot értelmezzük, amit a Pareto-megoldásként értelmezünk. A feladatot így visszavezethetjük egy Holt-Modigliani-Muth-Simon (HMMS) problémára.

Kulcsszavak: Optimális irányítás, Ellátási láncok koordinációja, Nagykereskedelmi árszerződés, Dinamikus játék

Abstract.
We investigate a supply chain with a supplier and a manufacturer. The manufacturer knows the demand for her product. The product is produced from a product supplied by the supplier. The purchased product is stored in a cross-docking way, i.e. there is no inventory of the purchased product and the planned production is ordered from the supplier. The costs of the manufacturer consist of quadratic inventory holding costs, quadratic production cost, and linear purchasing cost. It is assumed that the market price of the end product is known as well, so the sales of the producer are calculated. The linear purchasing cost is paid to the supplier. The goal of the manufacturer is maximize her cumulated profits. The sales of the supplier are the ordering cost of the manufacturer. The costs of the supplier are the quadratic manufacturing and inventory holding costs. The goal of the supplier is to maximize the sales reduced with the relevant costs. The supplier and the manufacturer want to negotiate about the sales price of the supplier and the quantity ordered by the manufacturer. In this paper we will not examine the bargaining process which determines the adequate price and quantity. The situation is modeled as a differential game. The decision variables of the supplier are the sales price and the production quantity of the supplier. The manufacturer will choose the cost minimal production plan to minimize her costs, so maximize the cumulated profits. The problem is a differential game with two players. The basic problem is a Holt-Modigliani-Muth-Simon (HMMS) problem extended with linear purchasing costs. We will examine two cases: the decentralized Nash-solution and a centralized Pareto-solution to optimize the behaviors of the players of the game.

Keywords: Optimal control, Supply chain coordination, Wholesale price contract, Dynamic game

1. Introduction

Supply chain coordination is a tool in supply chain to avoid extra costs of the participants in the value chain. It is often called as vertical integration. There are a number of tools which lead to supply chain or channel coordination. The most important tool of the supply chain coordination is the contracts. (See Tsay et al. (1999) and Cachon (2003).) With supply contracts the members of a supply chain can avoid or reduce the well-known bullwhip effect. In this paper we examine one of the supply chain contracts: the wholesale price contract in a dynamic environment.

We investigate a supply chain with a supplier and a manufacturer. The manufacturer knows the demand for her product. The product is produced from a product supplied by the supplier. The purchased product is stored in a cross-docking way, i.e. there is no inventory of the purchased product and the planned production is ordered from the supplier. The costs of the manufacturer consist of quadratic inventory holding costs, quadratic production cost, and linear purchasing cost. It is assumed that the market price of the end product is known as well, so the sales of the producer are calculated. The linear purchasing cost is paid to the supplier. The goal of the manufacturer is maximize her cumulated profits. The sales of the supplier are the ordering cost of the manufacturer. The costs of the supplier are the quadratic manufacturing and inventory holding costs. The goal of the supplier is to maximize the sales reduced with the relevant costs. The basic problem was initiated by Holt et al. (1960). A similar problem was analyzed by Dobos (2003) in a HMMS-environment for a reverse logistics problem. The bullwhip effect was examined by Dobos (2010) in a HMMS-supply chain.

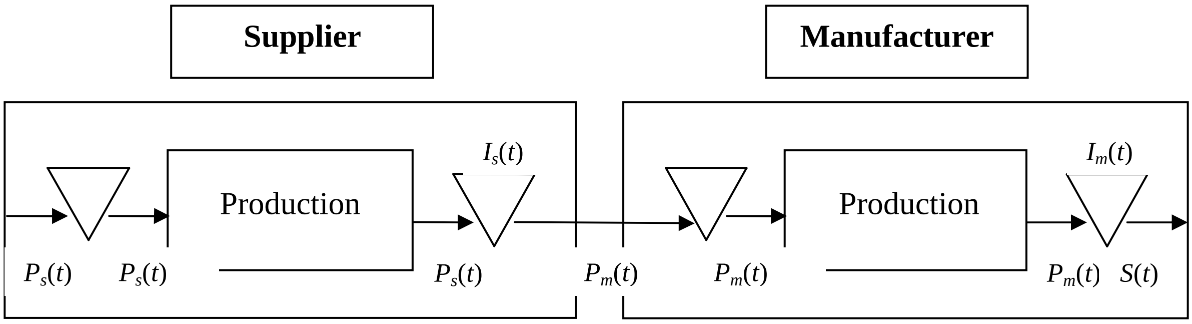
The supplier and the manufacturer want to negotiate about the sales (purchasing) price of the supplier and the quantity ordered by the manufacturer. The situation is modeled as a differential game. The decision variables of the supplier are the sales price and the production quantity of the supplier. The manufacturer will choose the cost minimal production plan to minimize her costs, so maximize the cumulated profits. The problem is a differential game with two players. The problem is a Holt-Modigliani-Muth-Simon (HMMS-) type model extended with linear purchasing costs. We will examine two cases: the decentralized Nash-solution and a centralized Pareto-solution to optimize the behaviors of the players of the game.

The paper is organized as follows. Section two presents the model with assumptions and solves the decentralized case with the concept of the differential game. The next chapter investigates the cooperative solution, i.e. the participants of the supply chain sums up all of the relevant costs and optimize the system-wide problem. In this section we will not investigate the sharing of the benefits. In section four we demonstrate the functioning of the models with a numerical example. Last we summarize the results of the paper.

2. The decentralized system: The Nash solution

We consider a simple supply chain consisting of two firms: a supplier and a manufacturer. We assume that the firms are independent, that is, each makes her decision to minimize her own costs. The firms have two stores: a store for raw materials and a store for end products. Moreover, we assume that the input stores are empty, that is, the firms can order suitable quantity and that they can get the ordered quantity. The production processes have a known, constant lead time. The material flow of the model is depicted in Figure 1.

Figure 1. Material flow in the models



The following parameters are used in the models:

T	length of the planning horizon,
$S(t)$	the rate of demand, continuous differentiable, $t \in [0,T]$,
$\bar{I}_m(t)$	inventory goal size of manufactured product, $t \in [0,T]$,
$\bar{I}_s(t)$	inventory goal size of supplied product, $t \in [0,T]$,
$\overline{P}_m(t)$	manufacturing goal level, $t \in [0,T]$,
$\overline{P}_s(t)$	supply goal level, $t \in [0,T]$,
h_m	inventory holding cost coefficient in manufactured product store,
h_s	inventory holding cost coefficient in supplied product store,
c_m	production cost coefficient for manufacturing,
c_s	production cost coefficient for supply,
p	market price of the product of the manufacturer.

In the HMMS-model it is assumed that the management of the (manufacturer and supplier) firms have fixed a production-inventory pattern, that is, the production plans $\overline{P}_m(t)$ and $\overline{P}_s(t)$, and planned inventory levels $\bar{I}_m(t)$ and $\bar{I}_s(t)$ are known before the planning horizon. The objective of the managers of the firms is to minimize the deviations from the fixed objective level. The deviations are defined, as quadratic functional with known parameters. This phenomenon was empirically tested by Holt, Modigliani, Muth, and Simon (1960).

The decision variables:

$I_m(t)$	the inventory level of the manufactured product, it is non-negative, $t \in [0,T]$,
$I_s(t)$	the inventory level of the supplied product, it is non-negative, $t \in [0,T]$,
$P_m(t)$	the rate of manufacturing, it is non-negative, $t \in [0,T]$,
$P_s(t)$	the rate of supply, it is non-negative, $t \in [0,T]$,
w	the purchasing price of the supplied product.

The decentralized model describes the situation where the supplier and the manufacturer optimize independently, we mean the manufacturer determines its optimal production-inventory strategy first (the market demand is given exogenously), then she orders the necessary quantity of products to meet the known demand. Then the supplier accepts the order and minimizes her own costs.

Next, we model the manufacturer in this HMMS-environment. The manufacturer solves the following problem for a given purchasing price w^N:

$$J_m\left(P_m(.);w^N\right)=$$
$$\int_0^T p\cdot S(t)dt-\int_0^T\left\{\frac{h_m}{2}\left[I_m(t)-\bar{I}_m(t)\right]^2+\frac{c_m}{2}\left[P_m(t)-\bar{P}_m(t)\right]^2+w^N\cdot P_m(t)\right\}dt\rightarrow\max \qquad (1)$$

s.t.

$$\dot{I}_m(t)=P_m(t)-S(t),\ I_m(0)=I_{m0},\quad 0\le t\le T\ . \qquad (2)$$

The functional (1) is the cumulated profit of the manufacturer. It is easy to see that the revenue is a known constant, so the problem can be reformulated, as a cost minimization in the next way:

$$J'_m\left(P_m(.);w^N\right)=\int_0^T\left\{\frac{h_m}{2}\left[I_m(t)-\bar{I}_m(t)\right]^2+\frac{c_m}{2}\left[P_m(t)-\bar{P}_m(t)\right]^2+w^N\cdot P_m(t)\right\}dt\rightarrow\min\,. \qquad (1')$$

Assume that the optimal production-inventory policy of the manufacturer in dependence of the purchasing price w is $\left(I_m^N(\cdot),P_m^N(\cdot)\right)$ in model (1')-(2) and the manufacturer orders $P_m^N(\cdot)$. Then the supplier solves the following problem:

$$J_s\left(P_s(.),w;P_m^N(.)\right)=w\cdot\int_0^T P_m^N(t)dt-\int_0^T\left\{\frac{h_s}{2}\left[I_s(t)-\bar{I}_s(t)\right]^2+\frac{c_s}{2}\left[P_s(t)-\bar{P}_s(t)\right]^2\right\}dt\rightarrow\min \qquad (3)$$

s.t.

$$\dot{I}_s(t)=P_s(t)-P_m^N(t),\ I_s(0)=I_{s0},\quad 0\le t\le T \qquad (4)$$

Notice that problem (3)-(4) has the same planning horizon [0,T] as that of model (1')-(2). To solve problem (1')-(2) we apply the Pontryagin's Maximum Principle (see e.g. Feichtinger and Hartl, (1986), Seierstad and Sydsaeter, (1987)). The Hamiltonian function of this problem is as follows:

$$H_m\left(I_m(t),P_m(t),\psi_m(t),t\right)=$$
$$-\left\{\frac{h_m}{2}\left[I_m(t)-\bar{I}_m(t)^2\right]+\frac{c_m}{2}\left[P_m(t)-\bar{P}_m(t)\right]^2+w^N\cdot P_m(t)\right\}+\psi_m(t)\cdot\left(P_m(t)-D(t)\right).$$

This problem is an optimal control problem with pure state variable constraints. To obtain the necessary and sufficient conditions of optimality we need the Lagrangian function:

$$L_m\left(I_m(t),P_m(t),\psi_m(t),\lambda_m(t),t\right)=H_m\left(I_m(t),P_m(t),\psi_m(t),t\right)+\lambda_m(t)\cdot I_m(t)\ .$$

Lemma 1 $\left(I_m^N(\cdot), P_m^N(\cdot)\right)$ is the optimal solution of problem (1')-(2) if and only if there exists continuous function $\psi_m(\cdot)$ such that for all $0 \leq t \leq T$ $\psi_m(t) \neq 0$ and

(a) $$\dot{\psi}_m(t) = -\frac{\partial L_m\left(I_m^N(t), P_m^N(t), \psi_m(t), \lambda_m(t), t\right)}{\partial I_m(t)} \quad \Rightarrow \quad \dot{\psi}_m(t) = h_m\left[I_m^N(t) - \bar{I}_m(t)\right] - \lambda_m(t)$$

(b)
$$\max_{P_m(t)\geq 0}\left\{H_m\left(I_m^N(t), P_m(t), \psi_m(t), t\right)\right\} = H_m\left(I_m^N(t), P_m^N(t), \psi_m(t), t\right)$$
$$\Rightarrow \quad \max_{P_m(t)\geq 0}\left\{H_m\left(I_m^N(t), P_m(t), \psi_m(t), t\right)\right\} = \left[\psi_m(t) - w^N\right] \cdot P_m^N(t) - \frac{c_m}{2}\left[P_m^N(t) - \bar{P}_m(t)\right]^2,$$

(c) $$\lambda_m(t) \cdot I_m^N(t) = 0, \quad \lambda_m(t) \geq 0,$$

(d) $$\psi_m(T) \cdot I_m^N(T) = 0, \quad \psi_m(T) \geq 0.$$

We do not prove the above lemma; its proof can be found in the above mentioned literature. The optimal solution can be easy constructed, if the optimal production rate and the optimal inventory level are positive in along the planning horizon.

Lemma 2 Assume that production-inventory strategy $\left(I_m^N(\cdot), P_m^N(\cdot)\right)$ is an optimal solution for model (1')-(2). Then the optimal solution must satisfy the following differential equation:

$$\begin{pmatrix} \dot{I}_m^N(t) \\ \dot{P}_m^N(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{h_m}{c_m} & 0 \end{pmatrix} \cdot \begin{pmatrix} I_m^N(t) \\ P_m^N(t) \end{pmatrix} + \begin{pmatrix} -S(t) \\ \dot{\bar{P}}_m(t) - \frac{h_m}{c_m} \cdot \dot{\bar{I}}_m(t) \end{pmatrix},$$

with initial and terminal condition

$$I_m^N(0) = I_{m0}, \; P_m^N(T) = \bar{P}_m(T) - \frac{w^N}{c_m}.$$

We do not prove this lemma, the proof can be found in Dobos (2003). If production strategy $P_m^N(\cdot)$ is known, then problem (3)-(4) can be solved.

The manufacturer will order from the supplier, if the cumulated profit is nonnegative in the planning horizon in dependence of the purchasing price, i.e.

$$\int_0^T p \cdot S(t)dt - \int_0^T \left\{\frac{h_m}{2}\left[I_m^N(t) - \bar{I}_m(t)\right]^2 + \frac{c_m}{2}\left[P_m^N(t) - \bar{P}_m(t)\right]^2 + w^N \cdot P_m(t)\right\}dt \geq 0.$$

If we solve the problem (1)-(2) changing the purchasing price, we achieve an upper bound for this price $0 \leq w^N \leq \bar{w}$. This means that the optimal cumulated profit of the manufacturer is even zero, if the purchasing price is equal to the upper bound $\bar{w}$.

After optimal production strategy $P_m^N(\cdot)$ is given we can solve problem (3)-(4). The Hamiltonian function of problem (3)-(4) is as follows

$$H_s\left(I_s(t), P_s(t), \psi_s(t), t\right) = -\left\{\frac{h_s}{2}\left[I_s(t) - \bar{I}_s(t)^2\right] + \frac{c_s}{2}\left[P_s(t) - \overline{P}_s(t)\right]^2\right\} + \psi_s(t) \cdot \left(P_s(t) - P_m^N(t)\right).$$

This problem is also an optimal control problem with pure state variable constraints. To get the necessary and sufficient conditions of optimality, we need again the Lagrangian function:

$$L_s\left(I_s(t), P_s(t), \psi_s(t), \lambda_s(t), t\right) = H_s\left(I_s(t), P_s(t), \psi_s(t), t\right) + \lambda_s(t) \cdot I_s(t).$$

The proof of the following lemma can be found again in the mentioned literature.

Lemma 3 $\left(I_s^N(\cdot), P_s^N(\cdot), w^N\right)$ is optimal solution of problem (3)-(4), if and only if there exists continuous function $\psi_s(\cdot)$ such that for all $0 \le t \le T$ $\psi_s(t) \neq 0$ and

(a) $\quad \dot{\psi}_s(t) = -\dfrac{\partial L_s\left(I_s^N(t), P_s^N(t), \psi_s(t), \lambda_s(t), t\right)}{\partial I_s(t)} \quad \Rightarrow \quad \dot{\psi}_s(t) = h_s\left[I_s^N(t) - \bar{I}_s(t)\right] - \lambda_s(t),$

(b) $\quad \max\limits_{P_s(t) \ge 0}\left\{H_s\left(I_s^N(t), P_s(t), \psi_s(t), t\right)\right\} = H_s\left(I_s^N(t), P_s^N(t), \psi_s(t), t\right)$

$\quad \Rightarrow \quad \max\limits_{P_s(t) \ge 0}\left\{H_s\left(I_s^N(t), P_s(t), \psi_s(t), t\right)\right\} = \psi_s(t) \cdot P_s^N(t) - \dfrac{c_s}{2}\left[P_s^N(t) - \overline{P}_s(t)\right]^2,$

(c) $\quad \lambda_s(t) \cdot I_s^d(t) = 0, \quad \lambda_s(t) \ge 0,$

(d) $\quad \psi_s(T) \cdot I_s^N(T) = 0, \quad \psi_s(T) \ge 0,$

(e) $\quad w^N = \overline{w}.$

For the case of positive inventory level and production rate the optimal strategy is presented in the next lemma.

Lemma 4 Let us assume that production-inventory strategy $\left(I_s^N(\cdot), P_s^N(\cdot)\right)$ is an optimal solution for model (3)-(4). Then the optimal solution must satisfy the following differential equation:

$$\begin{pmatrix} \dot{I}_s^N(t) \\ \dot{P}_s^N(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \dfrac{h_s}{c_s} & 0 \end{pmatrix} \cdot \begin{pmatrix} I_s^N(t) \\ P_s^N(t) \end{pmatrix} + \begin{pmatrix} -P_m^N(t) \\ \dot{\overline{P}}_s(t) - \dfrac{h_s}{c_s} \cdot \dot{\bar{I}}_s(t) \end{pmatrix},$$

with initial and terminal condition

$$I_s^N(0) = I_{s0},\ P_s^N(T) = \overline{P}_s(T).$$

Later we use the following notations: let J_m^N and J_s^N be the optimal values of cost functions (1) and (3) respectively, that is, let

$$J_m^N = \int_0^T \left\{ \frac{h_m}{2} \left[I_m^N(t) - \bar{I}_m(t) \right]^2 + \frac{c_m}{2} \left[P_m^N(t) - \bar{P}_m(t) \right]^2 + w^N \cdot P_m^N(t) \right\} dt$$

and

$$J_s^N = \int_0^T \left\{ \frac{h_s}{2} \left[I_s^N(t) - \bar{I}_s(t) \right]^2 + \frac{c_s}{2} \left[P_s^N(t) - \bar{P}_s(t) \right]^2 - w^N \cdot P_m^N(t) \right\} dt .$$

The solutions of the two models are the well-known Nash solution of this game model. This connection can be written, as

$$J_m^N = J_m\left(P_m^N(.); w^N\right) \leq J_m\left(P_m(.); w^N\right)$$

and

$$J_s^N = J_s\left(P_s^N(.), w_s^N; P_s^N(.)\right) \leq J_s\left(P_s(.), w; P_s^N(.)\right).$$

In the next section we investigate the coordinated solution of the examined model. We assume that the supplier and manufacturer act as a common firm and minimize the costs together. The Nash solution has a property that the profit of the manufacturer is equal to zero. It means that all of the profits are realized at the supplier in our model.

3 The centralized system: The Pareto optimal solution

In this section we solve the centralized model, that is, the model, where the manufacturer and supplier coordinate their decisions and sum up the relevant profits. The cost functional does not depend on the purchasing price in this model. The purchasing price is an inner accounting tool of the gains. In this paper we do not investigate the distribution of the profits at the end of the planning horizon. The model is as follows

$$J_{ms}(P_m(.), P_s(.)) =$$
$$\int_0^T \left\{ \frac{h_m}{2} \left[I_m(t) - \bar{I}_m(t) \right]^2 + \frac{c_m}{2} \left[P_m(t) - \bar{P}_m(t) \right]^2 + \frac{h_s}{2} \left[I_s(t) - \bar{I}_s(t) \right]^2 + \frac{c_s}{2} \left[P_s(t) - \bar{P}_s(t) \right]^2 \right\} dt \to \min \tag{5}$$

s.t.

$$\dot{I}_m(t) = P_m(t) - S(t), \quad 0 \leq t \leq T \tag{6}$$

$$\dot{I}_s(t) = P_s(t) - P_m(t), \quad 0 \leq t \leq T, \tag{7}$$

$$\begin{pmatrix} I_m(0) \\ I_s(0) \end{pmatrix} = \begin{pmatrix} I_{m0} \\ I_{s0} \end{pmatrix} \tag{8}$$

The Hamiltonian function of model (5)-(8) is

$$H(I_m(t), P_m(t), I_s(t), P_s(t), \psi_m(t), \psi_s(t))$$
$$= -\left\{ \frac{h_m}{2}\left[I_m(t) - \bar{I}_m(t)\right]^2 + \frac{c_m}{2}\left[P_m(t) - \overline{P}_m(t)\right]^2 + \frac{h_s}{2}\left[I_s(t) - \bar{I}_s(t)\right]^2 + \frac{c_s}{2}\left[P_s(t) - \overline{P}_s(t)\right]^2 \right\}$$
$$+\psi_m(t)\cdot\left[P_m(t) - S(t)\right] + \psi_s(t)\cdot\left[P_s(t) - P_m(t)\right] .$$

The Lagrangian function is

$$L\left(I_m(t), P_m(t), I_s(t), P_s(t), \psi_m(t), \psi_s(t), \lambda_m(t), \lambda_s(t), t\right)$$
$$= H\left(I_m(t), P_m(t), I_s(t), P_s(t), \psi_m(t), \psi_s(t), t\right) + \lambda_m(t)\cdot I_m(t) + \lambda_s(t)\cdot I_s(t) .$$

The following lemma formalizes the well-known optimality conditions. Its proof can be found in the literature mentioned in the previous section.

Lemma 5 $\left(I_m^P(\cdot), P_m^P(\cdot), I_s^P(\cdot), P_s^P(\cdot)\right)$ is optimal solution of problem (5)-(8), if and only if the following points hold

a)
$$\frac{\partial L\left(I_m^P(t), P_m^P(t), I_s^P(t), P_s^P(t), \psi_m(t), \psi_s(t), \lambda_m(t), \lambda_s(t), t\right)}{\partial I_m(t)}$$
$$= -h_m\left[I_m^P(t) - \bar{I}_m(t)\right] + \lambda_m(t) = -\dot{\psi}_m(t),$$

$$\frac{\partial L\left(I_m^P(t), P_m^P(t), I_s^P(t), P_s^P(t), \psi_m(t), \psi_s(t), \lambda_m(t), \lambda_s(t), t\right)}{\partial I_s(t)}$$
$$= -h_s\left[I_s^P(t) - \bar{I}_s(t)\right] + \lambda_s(t) = -\dot{\psi}_s(t),$$

b)
$$\max_{P_m(t)\geq 0}\left\{H\left(I_m^P(t), P_m(t), I_s^P(t), P_s^P(t), \psi_m(t), \psi_s(t), t\right)\right\}$$
$$= \left(\psi_m(t) - \psi_s(t)\right)\cdot P_m^P(t) - \frac{c_m}{2}\left[P_m^P(t) - \overline{P}_m(t)\right]^2,$$

$$\max_{P_s(t)\geq 0}\left\{H\left(I_m^P(t), P_m(t), I_s^P(t), P_s^P(t), \psi_m(t), \psi_s(t), t\right)\right\}$$
$$= \psi_s(t)\cdot P_s^P(t) - \frac{c_s}{2}\left[P_s^P(t) - \overline{P}_s(t)\right]^2,$$

c) $\lambda_m(t)\cdot I_m^P(t) = 0, \quad \lambda_m(t) \geq 0$,

$\lambda_s(t)\cdot I_s^P(t) = 0, \quad \lambda_s(t) \geq 0$,

d) $\left(\psi_m(T) - \psi_s(T)\right)\cdot I_m^P(T) = 0, \quad \psi_m(T) \geq 0.$

$$\psi_s(T)\cdot I_s^P(T)=0, \quad \psi_s(T)\geq 0.$$

The optimal centralized production strategies for the manufacturer and the supplier respectively are

$$P_m^P(t)=\begin{cases} 0, & \text{if } \dfrac{\psi_m(t)-\psi_s(t)}{c_m}+\bar{P}_m(t)\leq 0, \\ \dfrac{\psi_m(t)-\psi_s(t)}{c_m}+\bar{P}_m(t), & \text{if } \dfrac{\psi_m(t)-\psi_s(t)}{c_m}+\bar{P}_m(t)>0, \end{cases}$$

and

$$P_s^P(t)=\begin{cases} 0, & \text{if } \dfrac{\psi_s(t)}{c_c}+\bar{P}_s(t)\leq 0, \\ \dfrac{\psi_s(t)}{c_c}+\bar{P}_s(t), & \text{if } \dfrac{\psi_s(t)}{c_c}+\bar{P}_s(t)>0, \end{cases}$$

These two equations are the *optimal linear decision rules*. (See Holt-Modigliani-Muth-Simon (1960).) Differentiating adjoint variables $\psi_m(\cdot)$ and $\psi_s(\cdot)$, and then substituting into the conditions, the necessary and sufficient conditions become a system of linear differential equations:

$$\begin{pmatrix} \dot{I}_m^P(t) \\ \dot{I}_s^P(t) \\ \dot{P}_m^P(t) \\ \dot{P}_s^P(t) \end{pmatrix}=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ \dfrac{h_m}{c_m} & -\dfrac{h_s}{c_m} & 0 & 0 \\ 0 & \dfrac{h_s}{c_s} & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} I_m^P(t) \\ I_s^P(t) \\ P_m^P(t) \\ P_s^P(t) \end{pmatrix}+\begin{pmatrix} -S(t) \\ 0 \\ \dot{\bar{P}}_m(t)-\dfrac{h_m}{c_m}\bar{I}_m(t)+\dfrac{h_s}{c_m}\bar{I}_s(t) \\ \dot{\bar{P}}_s(t)-\dfrac{h_s}{c_s}\bar{I}_s(t) \end{pmatrix} \tag{9}$$

with initial and ending conditions

$$\begin{pmatrix} I_m^P(0) \\ I_s^P(0) \end{pmatrix}=\begin{pmatrix} I_{m0} \\ I_{s0} \end{pmatrix},$$

and

$$\begin{pmatrix} P_m^P(T) \\ P_s^P(T) \end{pmatrix}=\begin{pmatrix} \bar{P}_m(T) \\ \bar{P}_s(T) \end{pmatrix}.$$

Finally, consider a notation: let $J_{ms}^P=J_{ms}\left(P_m^P(.),P_s^P(.)\right)$ denote the optimal value of cost function (5). It is easy to see that $J_{ms}^P\leq J_m^N+J_s^N$, i.e. the Pareto solution of the problem has a lower cost and a higher profit.

4. A numerical example

Take the following parameters and cost functions in problems (1)-(2), (3)-(4) and (5)-(8), as in Table 1.

Table 1. Parameter specification for the example

Description	**Data**
Length of planning horizon: T	5
Demand rates: $S(t)$	$\sin(t)+2$
Delay of the supply: τ	0.5
Manufacturing rate goal level: $\overline{P}_m(\cdot)$	1.0
Supply rate goal level: $\overline{P}_s(\cdot)$	0.85
Inventory size goal level in manufacturing store: $\bar{I}_m(\cdot)$	0.5
Inventory size goal level in supply store: $\bar{I}_s(\cdot)$	0.3
Initial inventory level in manufacturing store: $I_m(0)$	0.25
Initial inventory level in manufacturing store: $I_s(0)$	0.5
Manufacturing cost coefficient: c_m	1.0
Supply cost coefficient: c_s	0.5
Inventory holding cost coefficient in manufacturing store: h_m	2
Inventory holding cost coefficient in supply store: h_s	1
Sales price of the end product	0.764

In the following we solve the decentralized and the centralized problem.

5.1 The solution of the decentralized problem

The decentralized problem is a hierarchical production planning problem. First the manufacturer solves her planning problem then the optimal ordering policy is forwarded to the supplier. Finally, the supplier optimizes her own relevant costs based on the known ordering policy of the manufacturer.

First we determine the upper bound for the purchasing price. If the purchasing price is greater than this value, then the manufacturer has a negative profit and she looks for a new supplier with a lower purchasing price. The problem of the manufacturer is as follows:

$$J_m(w) = \int_0^5 \left\{ \frac{2}{2}[I_m(t) - 0.5]^2 + \frac{1}{2}[P_m(t) - 1]^2 + w \cdot P_m(t) \right\} dt \to \min$$

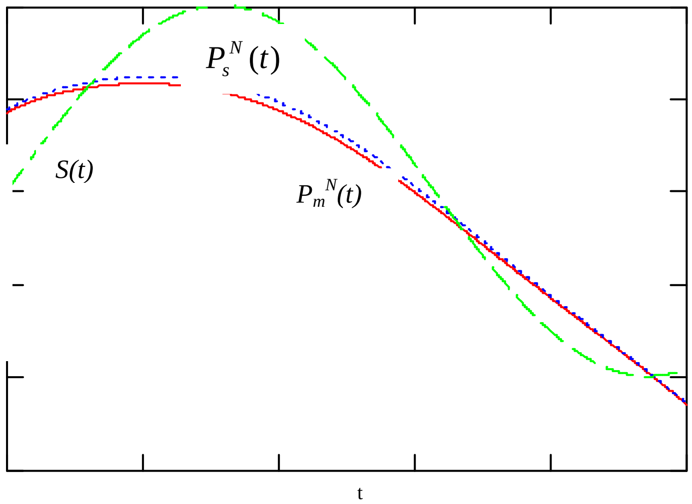
s.t.

$$\dot{I}_m(t) = P_m(t) - \sin(t) - 2, \; I_m(0) = 0.25, \quad 0 \le t \le 5$$

The revenue of the manufacturer is equal to 12.56. The upper bound is 0.4, i.e. $\overline{w} = 0.4$. Let us now substitute this value of the cost function of the manufacturer. The optimal solution can be determined with help of Lemma 2, because the optimal inventory level and production rate are positive. Let the optimal the optimal solution be functions $P_m^N(\cdot)$ and $I_m^N(\cdot)$.

The minimal cost of the manufacturer is 9.594 units, that is, $J_m^N = 9.594$. The cumulate profit is zero.

Figure 2. Optimal manufacturing and supply rates for the decentralized models



In the next step we solve the problem of the supplier, where the manufacturer's ordering policy $P_m^N(\cdot)$ is given:

$$J_s = \int_0^5 0.4 \cdot P_m^N(t)dt - \int_0^5 \left\{ \frac{1}{2}[I_s(t) - 0.3]^2 \frac{0.5}{2}[P_s(t) - 0.85]^2 \right\} dt \to \min$$

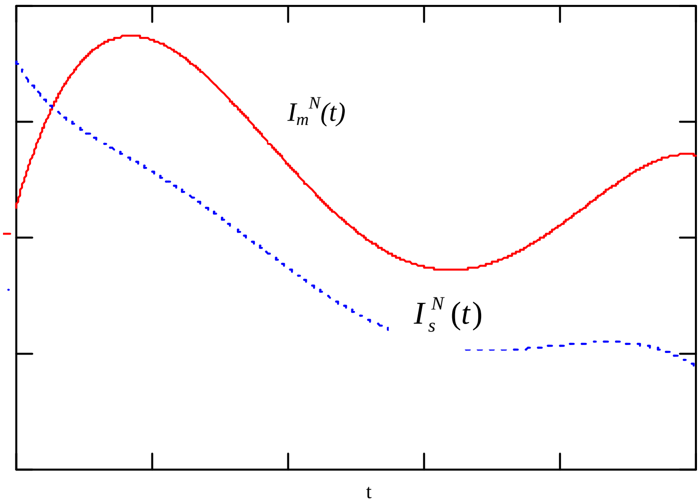
s.t.

$$\dot{I}_s(t) = P_s(t) - P_m^N(t), \ I_s(0) = 0.5, \quad 0 \le t \le 5$$

The optimal solution for the supplier is functions $P_s^N(\cdot)$ and $I_s^N(\cdot)$, applying the results of Lemma 4.

The optimal production rates and inventory levels are shown in Figures 2 and 3.

Figure 3. Optimal manufacturing and supply inventory levels for the decentralized models



The cumulated profit of the supplier is 1.306 units, that is $J_s^N = 1.306$. The total cost of manufacturer and supplier is 8.288 units in this decentralized strategy of the supply chain, that is $J_m^N + J_s^N = 8.288$.

5.2 The solution of the centralized problem

In the following we solve the centralized problem:

$$J_{ms} = \int_0^5 \left\{ \frac{2}{2}[I_m(t) - 0.5]^2 + \frac{1}{2}[P_m(t) - 1]^2 + \frac{1}{2}[I_s(t) - 0.3]^2 + \frac{0.5}{2}[P_s(t) - 0.85]^2 \right\} dt \to \min$$

s.t.

$$\dot{I}_m(t) = P_m(t) - \sin(t) - 2, \quad 0 \le t \le 5$$

$$\dot{I}_s(t) = P_s(t) - P_m(t), \quad 0 \le t \le 5$$

$$\begin{pmatrix} I_m(0) \\ I_s(0) \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.5 \end{pmatrix}$$

The optimal solution of this problem is given after the solution of the following differential equation (see (9)):

$$\begin{pmatrix} \dot{I}_m^P(t) \\ \dot{I}_s^P(t) \\ \dot{P}_m^P(t) \\ \dot{P}_s^P(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 2 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} I_m^P(t) \\ I_s^P(t) \\ P_m^P(t) \\ P_s^P(t) \end{pmatrix} + \begin{pmatrix} -\sin(t) - 2 \\ 0 \\ -0.7 \\ -0.6 \end{pmatrix} \qquad (9)$$

with initial and terminal conditions

$$\begin{pmatrix} I_m(0) \\ I_s(0) \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.5 \end{pmatrix},$$

and

$$\begin{pmatrix} P_m^P(T) \\ P_s^P(T) \end{pmatrix} = \begin{pmatrix} 1 \\ 0.85 \end{pmatrix}.$$

The optimal production rates and inventory levels are shown in Figures 4 and 5.

Figure 4. Optimal manufacturing and supply rates for the centralized model

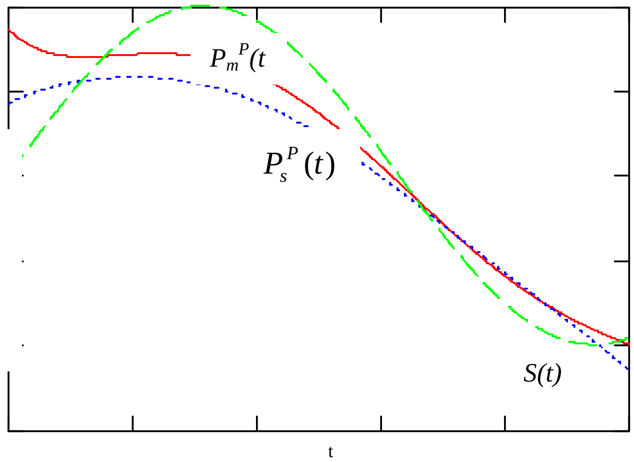
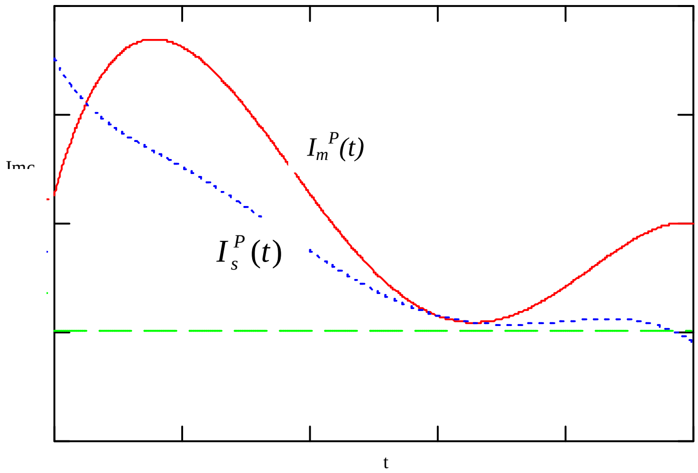


Figure 5. Optimal manufacturing and supply inventory levels for the centralized model



The minimal cost of the centralized system is 6.887 units, where the manufacturer's cost is 4.656 units and the supplier's cost is 2.231 units, that is, $J_{ms}^c = 6.887$, $J_m^c = 4.656$ and $J_s^c = 2.231$.

5.3 Comparison of the solutions of the decentralized and the centralized system

First, compare the production rate and inventory level of the manufacturer and the supplier in the cases of the decentralized and the centralized system, where Imd_t, Imc_t, Isd_t and Isc_t are for the inventory level for the manufacturer and for the supplier in the decentralized and the centralized model respectively.

Figure 6 The inventory level of the manufacturer in the decentralized and the centralized system

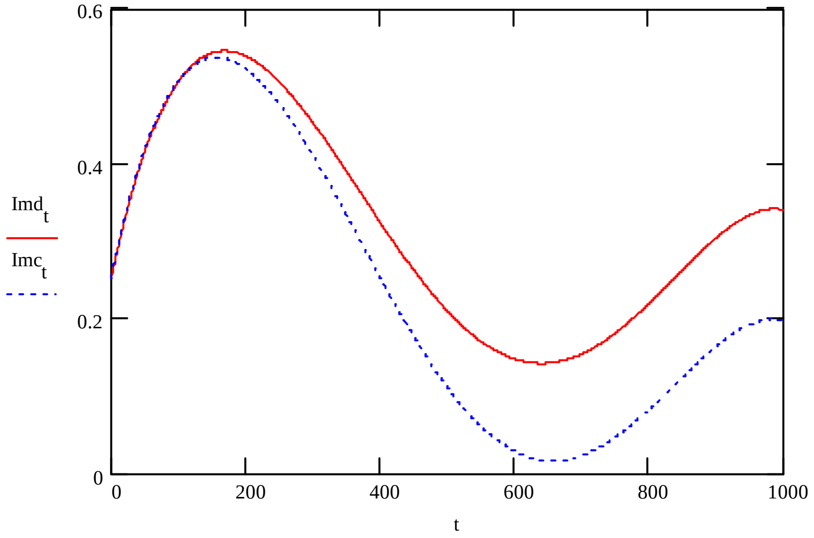
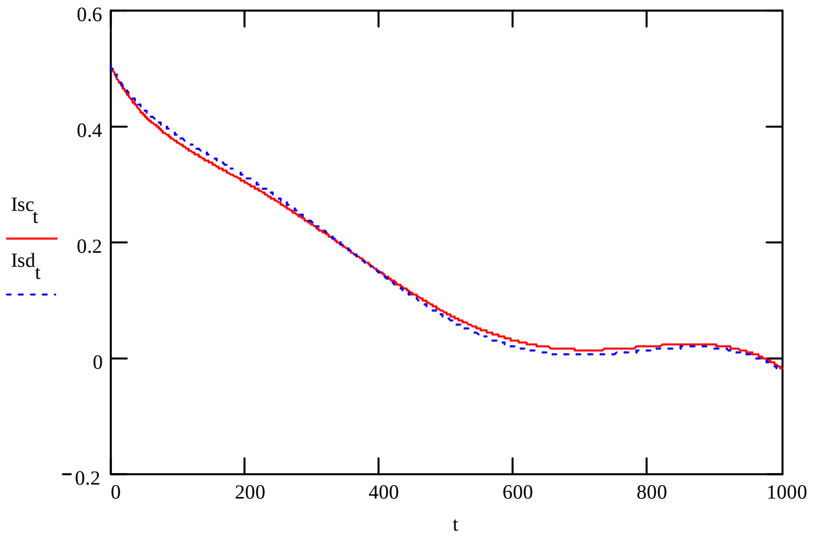
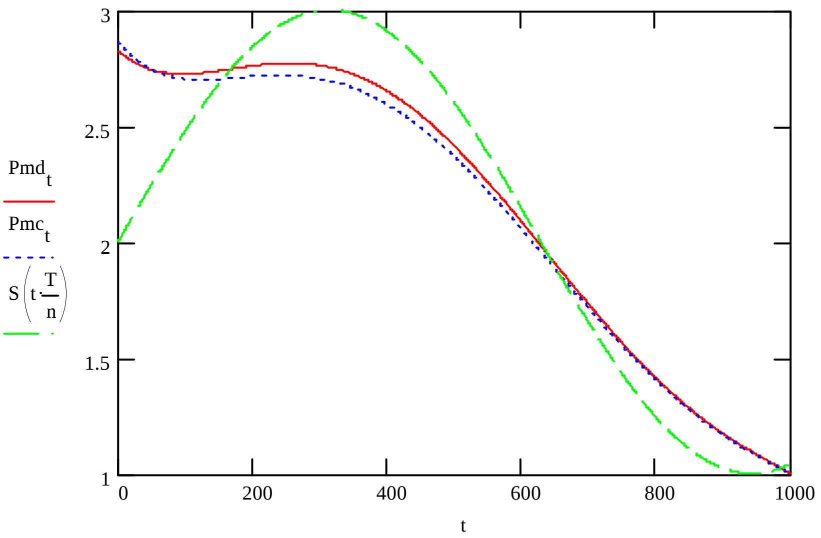


Figure 7 The inventory level of the supplier in the decentralized and the centralized system



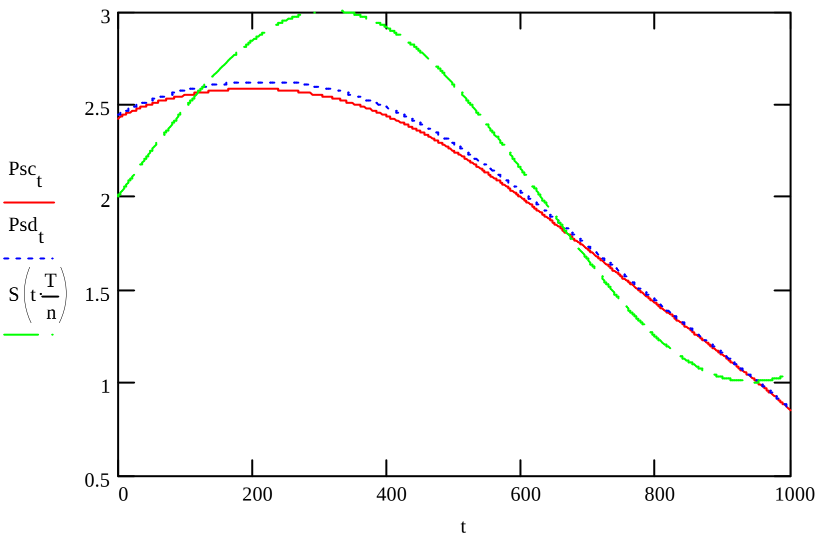
In this example the inventory level of the manufacturer decreases in the case of cooperation, that is, in the centralized system. The inventory level of the supplier first decreases, and then increases when the participants cooperate in the supply chain, see Figures 6 and 7.

Figure 8 The production rate of the manufacturer in the decentralized and the centralized system



As we see, the production level in the centralized system is smoother, that is, the growth of the production rate is smaller than that in the case of the decentralized system, and the contrary is true for the supplier, that is, in the decentralized system the production rate of the supplier is smoother than that in the centralized system, where Pmd_t, Pmc_t, Psd_t and Psc_t are for the production level for the manufacturer and for the supplier in the decentralized and the centralized models respectively, and S(t) is for the exogenously given demand, see Figures 8 and 5. This phenomenon is the decreased bullwhip effect in the centralized model.

Figure 9 The production rate of manufacturer in the decentralized and the centralized system



The optimal costs of the decentralized and the centralized problem are presented in Table 2.

Table 2. The optimal profits of the example

	Decentralized problem	**Centralized problem**
Manufacturer costs (J_m)	9.594	4.656
Supplier costs (J_s)	-1.306	2.231
Total costs (J_{ms})	8.288	6.887

As we have seen, the total cost of the centralized problem is lower than that of the decentralized one. The cost reduction is approximately 1%. In the centralized problem the manufacturer cost increases with more than 1% and the supplier cost decreases with 4.5%.

After the above analysis the question of how to share the savings, the cooperation of the participants in the supply chain induces, comes on stage.

5. Conclusion and further research

In this paper we have solved two two-stage HMMS-type supply chain models: a decentralized and a centralized model. We have showed that the cooperation of the two players induces savings in costs.

As an illustration for our results we have presented an exact number example. In this example the supplier's cost of adaption in production to the fluctuations in the orderings of the manufacturer is higher than that of the manufacturer. Moreover, the production costs are dominant over the inventory costs. Therefore it is not surprising at all that in the centralized model the supplier has reduced her inventory level, and the manufacturer's inventory level is higher than that in the decentralized model, and vice versa for the supplier.

The reason of this fact is that the manufacturer minimizes her relevant cost in the decentralized model, so that her production level is near to the demand rate. After cooperation the manufacturer gives up to follow her cost optimal production strategy to allow the supplier to reduce her own production-inventory cost implying a decrease in the total cost of the supply chain as well, since the supplier's cost saving balances out the increase of the manufacturer's cost.

This phenomenon points at the well known bullwhip effect of supply chains in a way: the supplier decreased the inventory level after information sharing (cooperation), and she adjusted her production rate closer to the demand rate.

In this type supply chains the two players might have asymmetrical roles. It can happen that the manufacturer has much stronger bargaining position than that of the supplier or vice versa. To analyze the mentioned bargaining situation is left for a further research.

References.

1. Cachon, G.P., “Supply Chain Coordination with Contracts” In de Kok, A.G & S.C. Graves (Eds), “Supply Chain Management: Design, Coordination and Operation,” *Elsevier*, : 2003, 229-339.
2. Dobos, I. (2003): Optimal production-inventory strategies for a HMMS-type reverse logistics system, Int. J. of Production Economics 81-82, 351-360.
3. Dobos, I. (2010): The analysis of bullwhip effect in a HMMS-type supply chain, International Journal of Production Economics, doi:10.1016/j.ijpe.2010.11.021
4. Feichtinger, G., Hartl, R.F. (1986): Optimale Kontrolle ökonomischer Prozesse: Anwendungen des Maximumprinzips in den Wirtschaftswissenschaften, de Gruyter, Berlin
5. Holt, C.C., Modigliani, F., Muth, J.F., Simon, H.A. (1960): Planning Production, Inventories, and Work Forces, Prentice-Hall, Englewood Cliffs, N.J.
6. Seierstad, A., Sydsaeter, K. (1987): Optimal control theory with economic applications, North-Holland, Amsterdam
7. Tsay, A.A., Nahmias, S., & N.Agrawal, “Modeling Supply Chain Contracts: A Review.” In: S.Tayur: R.Ganeshan & M.Magazine (Eds), “Quantitative Models in Supply Chain Management,” *Kluwer Academic Publishers, Boston* : 1998, 299-336.