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1093 Budapest, Fővám tér 8.
☎ (+36 1) 482-5566, Fax: 482-5567
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Imre Dobos

Miklós Pintér

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H-1093 Budapest
Hungary

Cooperation in supply chains: A cooperative game theoretic analysis

Imre Dobos[1]

Miklós Pintér[2]

[1] Corvinus University of Budapest, Institute of Business Economics, H-1093 Budapest, Fővám tér 8., Hungary, imre.dobos@uni-corvinus.hu.
[2] Corvinus University of Budapest, Department of Mathematics, H-1093 Budapest, Fővám tér 13-15., Hungary, miklos.pinter@uni-corvinus.hu.

Absztrakt.
A cikkben a kooperatív játékelmélet fogalmait alkalmazzuk egy ellátási lánc esetében. Az ostorcsapás-hatás elemeit egy beszállító-termelő ellátási láncban ragadjuk meg egy Arrow-Karlin típusú modellben lineáris készletezési és konvex termelési költség mellett. Feltételezzük, hogy mindkét vállalat minimalizálja a fontosabb költségeit. Két működési rendszert hasonlítunk össze: egy hierarchikus döntéshozatali rendszert, amikor először a termelő, majd a beszállító optimalizálja helyzetét, majd egy centralizált (kooperatív) modellt, amikor a vállalatok az együttes költségüket minimalizálják. A kérdés úgy merül fel, hogy a csökkentett ostorcsapás-hatás esetén hogyan osszák meg a részvevők ebben a transzferálható hasznosságú kooperatív játékban.

Kulcsszavak: Optimális irányítás, Ellátási lánc, Ostorcsapás-hatás, Kooperatív játékelmélet

Abstract.
In this paper we apply cooperative game theory concepts to analyze supply chains. The bullwhip effect in a two-stage supply chain (supplier-manufacturer) in the framework of the Arrow-Karlin model with linear-convex cost functions is considered. It is assumed that both firms minimize their relevant costs, and two cases are examined: the supplier and the manufacturer minimize their relevant costs in a decentralized and in a centralized (cooperative) way. The question of how to share the savings of the decreased bullwhip effect in the centralized (cooperative) model is answered by transferable utility cooperative game theory tools.

Keywords: Optimal control, Supply chain, Bullwhip effect, Cooperative game theory

1 Introduction

In the *supply chain* literature so far only *non-cooperative game theory* concepts were applied, see e.g. Kogan and Tapiero (2007) and Sethi at al. (2005). In this paper we analyze supply chains by *cooperative game theory* tools. Our main question is that how the manufacturer and the supplier should share the savings they achieve by harmonizing their production plans. We apply the following cooperative game theory concepts: the *core* (Gillies (1959)), the *stable set* (von Neumann and Morgenstern (1944)), the *Shapley value* (Shapley (1953)) and the *nucleolus* (Schmeidler (1969)) to answer the above question.

In order to demonstrate the efficiency of cooperating in a supply chain we consider the so called *bullwhip effect*. The bullwhip effect explains the fluctuations of sales (demand), manufacturing and supply. The bullwhip effect was first observed by Forrester (1961), later Lee et al. (1997) rediscovered this phenomenon. They mentioned four basic causes of the bullwhip effect:

- Forrester effect, or lead-times and demand signal processing,
- Burbidge effect, or order batching,
- Houlihan effect, or rationing and gaming,
- promotion effect, or price fluctuations.

These (new) names were introduced by Disney et al. (2003).

There are two basic models to investigate the decision processes of a firm: the Wagner-Whitin and the Arrow-Karlin model. Both models have a stock-flow identity and a cost function. The difference between them lies in the cost functions. The well-known lot sizing model of Wagner and Whitin (1958) assumes a concave cost function. The second basic model applies a convex cost function.

The basis of this investigation is the well-known Arrow-Karlin type dynamic production-inventory model (Arrow and Karlin, 1958). In this model the inventory holding cost is a linear function and the production cost is a non-decreasing and convex function of the production level. The latest empirical analysis, see Ghali (2003), shows that the convexity of the cost function is a reasonable assumption.

The main goal of this paper is to demonstrate that cooperative game theory tools can be applied to supply chain analysis. We consider an Arrow-Karlin-type two-stage supply chain and analyse whether the bullwhip effect appears in this model. To show that because of the bullwhip effect the cooperation of the manufacturer and the supplier induces savings, we develop two models: a decentralized and a centralized Arrow-Karlin-type supply chain model.

The decentralized model assumes that first the manufacturer solves her production planning problem (the market demand is given exogenously) and her ordering process is based on the optimal production plan. Then the supplier minimizes her costs on the basis of the ordering of the manufacturer. In the centralized model it is assumed that the participants of the supply chain cooperate, i.e. they minimize the sum of their costs.

In the next step we compare the production-inventory strategies and the costs of the manufacturer and supplier in the two models to show that the bullwhip effect can be reduced by cooperation (centralized model). This cooperation can be defined as a kind of information sharing between the parties of the supply chain.

Finally, we discuss the question of how the manufacturer and the supplier should share the savings their cooperation induces. At this point we use concepts of *transferable utility cooperative games*.

The paper is organized as follows. The decentralized model is discussed in Section 2. Section 3 analyzes the centralized (cooperative) supply chain model. In Section 4 we introduce some concepts of cooperative game theory and define supply chain (cooperative) games given by the models discussed in Sections 2 and 3. Moreover, we apply the above mentioned four solution concepts of transferable utility cooperative games to answer the question of how the manufacturer and the supplier should share the savings, the result of their cooperation. An exact number example is given in Section 5. The last section briefly concludes.

2 The decentralized system

We consider a simple supply chain consisting of two firms: a supplier and a manufacturer. We assume that the firms are independent, i.e. each makes her decision to minimize her own costs. The firms have two stores: a store for raw materials and a store for end products. Moreover, we assume that the input stores are empty, i.e. the firms can order suitable quantity and that they can get the ordered quantity. The production processes have a known, constant lead time. The material flow of the model is depicted in Figure 1.

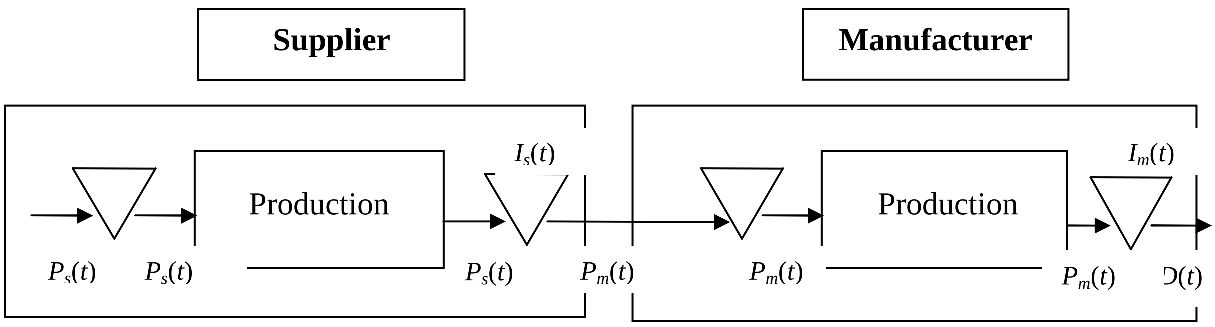

Figure 1 Material flow in the models

The following parameters are used in the models:

T	length of the planning horizon,
$D(t)$	the rate of demand, it is a continuous and differentiable function, $t \in [0,T]$,
h_m	the inventory holding coefficient in the manufacturer's product store,
h_s	the inventory holding coefficient in the supplier's product store,
$F_m(P_m(t))$	the production cost of the manufacturer at time t, it is a non-decreasing and strictly convex function,
$F_s(P_s(t))$	the production cost of the supplier at time t, it is a non-decreasing and strictly convex function.

The decision variables:

$I_m(t)$	the inventory level of the manufactured product, it is non-negative, $t \in [0,T]$,
$I_s(t)$	the inventory level of the supplied product, it is non-negative, $t \in [0,T]$,
$P_m(t)$	the rate of manufacturing, it is non-negative, $t \in [0,T]$,
$P_s(t)$	the rate of supply, it is non-negative, $t \in [0,T]$.

The decentralized model describes the situation where the supplier and the manufacturer optimize independently, we mean the manufacturer determines its optimal production-inventory strategy first (the market demand is given exogenously), then she orders the necessary quantity of products to meet the known demand. Then the supplier accepts the order and minimizes her own costs. The cost functions of the supplier and the manufacturer consist of two parts: the quadratic production costs and the inventory costs.

Next, we model the manufacturer in this Arrow-Karlin environment. The manufacturer solves the following problem:

$$J_m = \int_0^T \left[h_m \cdot I_m(t) + F_m\left(P_m(t)\right)\right] dt \to \min \tag{1}$$

s.t.

$$\dot{I}_m(t) = P_m(t) - D(t),\ I_m(0) = I_{m0}, \quad 0 \le t \le T \tag{2}$$

Assume that the optimal production-inventory policy of the manufacturer is $\left(I_m^d(\cdot), P_m^d(\cdot)\right)$ in model (1)-(2) and the manufacturer orders $P_m^d(\cdot)$. Then the supplier solves the following problem:

$$J_s = \int_0^T \left[h_s \cdot I_s(t) + F_s\left(P_s(t)\right)\right] dt \to \min \tag{3}$$

s.t.

$$\dot{I}_s(t) = P_s(t) - P_m^d(t),\ I_s(0) = I_{s0}, \quad 0 \le t \le T \tag{4}$$

Notice that problem (3)-(4) has the same planning horizon [0,*T*] as that of model (1)-(2).

To solve problem (1)-(2) we apply the Pontryagin's Maximum Principle (see e.g. Feichtinger and Hartl, (1986), Seierstad and Sydsaeter, (1987)). The Hamiltonian function of this problem is as follows:

$$H_m\left(I_m(t), P_m(t), \psi_m(t), t\right) = -\left[h_m \cdot I(t) + F_m\left(P_m(t)\right)\right] + \psi_m(t) \cdot \left(P_m(t) - S(t)\right).$$

This problem is an optimal control problem with pure state variable constraints. To obtain the necessary and sufficient conditions of optimality we need the Lagrangian function:

$$L_m\left(I_m(t), P_m(t), \psi_m(t), \lambda_m(t), t\right) = H_m\left(I_m(t), P_m(t), \psi_m(t), t\right) + \lambda_m(t) \cdot I_m(t).$$

Lemma 1 $\left(I_m^d(\cdot), P_m^d(\cdot)\right)$ is the optimal solution of problem (1)-(2) if and only if there exists continuous function $\psi_m(t)$ such that for all $0 \le t \le T$ $\psi_m(t) \ne 0$ and

(a)

$$\dot{\psi}_m(t) = -\frac{\partial L_m\left(I_m^d(t), P_m^d(t), \psi_m(t), \lambda_m(t), t\right)}{\partial I_m(t)} \quad \Rightarrow \quad \dot{\psi}_m(t) = h_m - \lambda_m(t),$$

(b)

$$\max_{P_m(t)\geq 0}\left\{H_m\left(I_m^d(t),P_m(t),\psi_m(t),t\right)\right\}=H_m\left(I_m^d(t),P_m^d(t),\psi_m(t),t\right)$$

$$\Rightarrow \quad \max_{P_m(t)\geq 0}\left\{H_m\left(I_m^d(t),P_m(t),\psi_m(t),t\right)\right\}=\psi_m(t)\cdot P_m^d(t)-F_m\left(P_m^d(t)\right),$$

(c) $\lambda_m(t)\cdot I_m^d(t)=0, \quad \lambda_m(t)\geq 0$,

(d) $\psi_m(T)\cdot I_m^d(T)=0, \quad \psi_m(T)\geq 0.$

We do not prove the above lemma, its proof can be found in the above mentioned literature. After optimal production strategy $P_m^d(\cdot)$ is given we can solve problem (3)-(4).

The Hamiltonian function of problem (3)-(4) is as follows

$$H_s\left(I_s(t),P_s(t),\psi_s(t),t\right)=-\left[h_s\cdot I(t)+F_s\left(P_s(t)\right)\right]+\psi_s(t)\cdot\left(P_s(t)-P_m^d(t)\right).$$

This problem is also an optimal control problem with pure state variable constraints. To get the necessary and sufficient conditions of optimality, we need again the Lagrangian function:

$$L_s\left(I_s(t),P_s(t),\psi_s(t),\lambda_s(t),t\right)=H_s\left(I_s(t),P_s(t),\psi_s(t),t\right)+\lambda_s(t)\cdot I_s(t).$$

The proof of the following lemma can be found again in the mentioned literature.

Lemma 2 $\left(I_s^d(\cdot),P_s^d(\cdot)\right)$ is optimal solution of problem (3)-(4) if and only if there exists continuous function $\psi_s(t)$ such that for all $0\leq t\leq T$ $\psi_s(t)\neq 0$ and

(a)

$$\dot{\psi}_s(t)=-\frac{\partial L_s\left(I_s^d(t),P_s^d(t),\psi_s(t),\lambda_s(t),t\right)}{\partial I_s(t)} \quad \Rightarrow \quad \dot{\psi}_s(t)=h_s-\lambda_s(t),$$

(b)

$$\max_{P_s(t)\geq 0}\left\{H_s\left(I_s^d(t),P_s(t),\psi_s(t),t\right)\right\}=H_s\left(I_s^d(t),P_s^d(t),\psi_s(t),t\right)$$

$$\Rightarrow \quad \max_{P_s(t)\geq 0}\left\{H_s\left(I_s^d(t),P_s(t),\psi_s(t),t\right)\right\}=\psi_s(t)\cdot P_s^d(t)-F_s\left(P_s^d(t)\right),$$

(c) $\lambda_s(t)\cdot I_s^d(t)=0, \quad \lambda_s(t)\geq 0$,

(d) $\psi_s(T)\cdot I_s^d(T)=0, \quad \psi_s(T)\geq 0.$

Later we use the following notations: let J_m^d and J_s^d be the optimal values of cost functions (1) and (3) respectively, i.e. let

$$J_m^d=\int_0^T\left[h_m\cdot I_m^d(t)+F_m\left(P_m^d(t)\right)\right]dt$$

and

$$J_s^d = \int_0^T \left[h_s \cdot I_s^d(t) + F_s\left(P_s^d(t)\right)\right]dt\ .$$

3 The centralized system

In this section we solve the centralized model, i.e. the model, where the manufacturer and supplier coordinate their decisions. The model is as follows

$$J_{ms} = \int_0^T \left[h_m \cdot I_m(t) + F_m\left(P_m(t)\right) + h_s \cdot I_s(t) + F_s\left(P_s(t)\right)\right]dt \to \min \tag{5}$$

s.t.

$$\dot{I}_m(t) = P_m(t) - D(t), \quad 0 \le t \le T \tag{6}$$

$$\dot{I}_s(t) = P_s(t) - P_m(t), \quad 0 \le t \le T, \tag{7}$$

$$\begin{pmatrix} I_m(0) \\ I_s(0) \end{pmatrix} = \begin{pmatrix} I_{m0} \\ I_{s0} \end{pmatrix} \tag{8}$$

The Hamiltonian function of model (5)-(8) is

$$H(I_m(t), P_m(t), I_s(t), P_s(t), \psi_m(t), \psi_s(t)) =$$
$$-h_m \cdot I_m(t) - F_m\left(P_m(t)\right) - h_s \cdot I_s(t) - F_s\left(P_s(t)\right) + \psi_m(t) \cdot \left[P_m(t) - S(t)\right] + \psi_s(t) \cdot \left[P_s(t) - P_m(t)\right].$$

The Lagrangian function is

$$L\left(I_m(t), P_m(t), I_s(t), P_s(t), \psi_m(t), \psi_s(t), \lambda_m(t), \lambda_s(t), t\right) =$$
$$= H\left(I_m(t), P_m(t), I_s(t), P_s(t), \psi_m(t), \psi_s(t), t\right) + \lambda_m(t) \cdot I_m(t) + \lambda_s(t) \cdot I_s(t).$$

The following lemma formalizes the well-known optimality conditions. Its proof can be found in the literature mentioned in the previous section.

Lemma 3 $\left(I_m^c(\cdot), P_m^c(\cdot), I_s^c(\cdot), P_s^c(\cdot)\right)$ is optimal solution of problem (5)-(8) if and only if the following points hold

1) $$\frac{\partial L\left(I_m^c(t), P_m^c(t), I_s^c(t), P_s^c(t), \psi_m(t), \psi_s(t), \lambda_m(t), \lambda_s(t), t\right)}{\partial I_m(t)} = -h_m + \lambda_m(t) = -\dot{\psi}_m(t)\ ,$$
$$\frac{\partial L\left(I_m^c(t), P_m^c(t), I_s^c(t), P_s^c(t), \psi_m(t), \psi_s(t), \lambda_m(t), \lambda_s(t), t\right)}{\partial I_s(t)} = -h_s + \lambda_s(t) = -\dot{\psi}_s(t)\ ,$$

2) $\max_{P_m(t)\geq 0}\left\{H\left(I_m^c(t), P_m(t), I_s^c(t), P_s^c(t), \psi_m(t), \psi_s(t), t\right)\right\} = \left(\psi_m(t) - \psi_s(t)\right)\cdot P_m^c(t) - F_m\left(P_m^c(t)\right),$

$\max_{P_s(t)\geq 0}\left\{H\left(I_m^c(t), P_m^c(t), I_s^c(t), P_s(t), \psi_m(t), \psi_s(t), t\right)\right\} = \psi_s(t)\cdot P_s^c(t) - F_s\left(P_s^c(t)\right),$

3) $\lambda_m(t)\cdot I_m^c(t) = 0, \quad \lambda_m(t)\geq 0,$

$\lambda_s(t)\cdot I_s^c(t) = 0, \quad \lambda_s(t)\geq 0,$

4) $\left(\psi_m(T) - \psi_s(T)\right)\cdot I_m^c(T) = 0, \quad \psi_m(T)\geq 0.$

$\psi_s(T)\cdot I_s^c(T) = 0, \quad \psi_s(T)\geq 0.$

The optimal centralized production strategies for the manufacturer and the supplier respectively are

$$P_m^c(t) = \begin{cases} 0, & if\ [F_m]^{-1}(\psi_m(t) - \psi_s(t)) \leq 0, \\ [F_m]^{-1}(\psi_m(t) - \psi_s(t)), & if\ [F_m]^{-1}(\psi_m(t) - \psi_s(t)) > 0, \end{cases}$$

and

$$P_s^c(t) = \begin{cases} 0, & if\ [F_m]^{-1}(\psi_s(t)) \leq 0, \\ [F_s]^{-1}(\psi_s(t)), & if\ [F_m]^{-1}(\psi_s(t)) > 0. \end{cases}$$

Finally, consider a notation: let $J_{ms}^c = J_m^c + J_s^c$ denote the optimal value of cost function (5), where

$$J_m^c = \int_0^T \left[h_m \cdot I_m^c(t) + F_m\left(P_m^c(t)\right)\right]dt$$

and

$$J_s^c = \int_0^T \left[h_s \cdot I_s^c(t) + F_s\left(P_s^c(t)\right)\right]dt.$$

4 The cooperative game theoretical solution of the cost sharing

In this section we provide a sharing rule of the savings the cooperation induces. It is easy to see the following result:

Lemma 4 $0 \leq J_{ms}^c = J_m^c + J_s^c \leq J_m^d + J_s^d$.

This result can be interpreted as follows: The total cost of the decentralized system, i.e. the sum of the supplier's and manufacturer's costs is higher than that of the centralized system. The question is now, how to share the savings induced by the players' cooperation.

First, we introduce the concept of transferable utility cooperative games. Let N= {1, 2,…, n} be the nonempty, finite set of the players. Moreover, let $v: 2^N \to \Re$ be a function such that $v(\emptyset) = 0$, where 2^N is for the class of all subsets of N. Then v is called transferable utility (TU) cooperative game, henceforth game with player set N.

Game v can be interpreted as every coalition (subset of N) has a value. E.g. $S \subseteq N$ is a coalition consisting of the players of S, and $v(S)$ is the value of coalition S. The value of a coalition can be the profit the coalition members can achieve if they cooperate, or the cost they induce if they harmonize their actions.

In our model there are two players: the manufacturer (m) and the supplier (s), i.e. $N = \{m, s\}$, and the value of a coalition is the cost the coalition member induce if they coordinate their production plans and inventory strategies.

In the decentralized model the players do not harmonize their actions, they achieve their minimal costs independently of each other. Therefore (see Subsection 4.1)

$$v(\{m\}) = J_m^d$$

and

$$v(\{s\}) = J_m^s .$$

In the centralized model the manufacturer and the supplier form a coalition, i.e. they cooperate. Therefore (see Subsection 4.2)

$$v(\{m, s\}) = J_{ms}^c .$$

Henceforth let v denote the supply chain game defined above.

To sum up the above discussion, the decentralized and the centralized model generate a (TU cooperative) game.

To answer the question of how the players should share the savings their cooperation induces, we apply four solution concepts of cooperative game theory.

First, we introduce the concept of *core* (Gillies (1959)). In our model the core of supply chain game v is defined as follows:

$$C(v) = \{x \in \Re^{\{m,s\}} : x_m + x_s = J_{ms}^c, x_m \le J_m^d, x_s \le J_s^d\},$$

where x_m and x_s are coordinates belonging to the manufacturer and the supplier respectively.

The core can be described as it consists of allocations of the total cost of the centralized model in the way of that none of the players can be better off by leaving the centralized model, by stopping cooperation, i.e. the core consists of stable (robust) allocations of the costs. It is easy to see that in this model *the core is not empty*, i.e. there is a stable allocation of the costs.

von Neumann and Morgenstern (1944) introduced the concept of *stable set*. The stable set, also called Neumann-Morgenstern solution. In our model the stable set is as follows:

Let $I(v) = \{x \in \Re^{\{m,s\}} : x_m + x_s = J_{ms}^c, x_m \le J_m^d, x_s \le J_s^d\}$, then $I(v)$ is called the set of imputations in supply chain game v. The stable set of supply chain game v, $S(v)$ is a subset of $I(v)$ such that

- inner stability: for any $x \in S(v)$, there does not exist $y \in S(v)$ such that $y_m + y_s < x_m + x_s$,
- outer stability: for all $x \in I(v) - S(v)$ there exists $y \in I(v)$ such that $y_m + y_s > x_m + x_s$.

The two stability conditions say that any element of the stable set cannot be better than any other point of the sable set, and for any imputation not in the stable set there exists an element of the stable set dominating the given imputation.

It is easy to see that in this model since $I(v) = C(v)$ and the two stability conditions are meaningless, we get the following result:

Lemma 5 Any supply chain game v has a unique stable set, and $S(v) = C(v)$.

Both the core and the stable set have the disadvantage that those generally consist of many points, i.e. those are map-valued solutions. Therefore, the following natural question comes up: How can we pick up only one point as a solution? Next we consider two point-valued solutions.

Shapley (1953) introduced the following point-valued solution concept: The *Shapley value* of the manufacturer and the supplier respectively in supply chain game v

$$Sh(v)_m = \frac{1}{2} J_m^d + \frac{1}{2}\left(J_{ms}^c - J_s^d\right),$$

and

$$Sh(v)_s = \frac{1}{2} J_s^d + \frac{1}{2}\left(J_{ms}^c - J_m^d\right).$$

The Shapley value can be interpreted as it is an expected value of the given player's marginal contribution. In other words, e.g. the manufacturer's Shapley value is the expected value with uniform distribution (1/2-1/2) of the manufacturer's marginal contribution to the cost of the two coalitions not containing her, to the empty collation (J_m^d) and to coalition $\{s\}$ ($J_{ms}^c - J_s^d$).

Next we show that in our model the Shapley solution is in the core and in the stable set, hence it is a real refinement of these two map-valued solution concepts.

Lemma 6 For any supply chain game v $\left(Sh(v)_m, Sh(v)_s\right) \in C(v)$.

Proof. Take the manufacturer first: Lemma 4 implies that

$$Sh(v)_m = \frac{1}{2} J_m^d + \frac{1}{2}(J_{ms}^c - J_s^d) \leq \frac{1}{2} J_m^d + \frac{1}{2} J_m^d,$$

i.e. $Sh(v)_m \leq J_m^d$. In a similar way we can see that $Sh(v)_s \leq J_s^d$.

Finally, it is well-known that $Sh(v)_m + Sh(v)_s = J_{ms}^c$ (see e.g. Shapley (1953)). □

Lemmata 5 and 6 imply that the Shapley solution of supply chain game v is in the stable set, i.e. $(Sh(v)_m, Sh(v)_s) \in S(v)$.

At last, we give the *nucleolus* of supply chain games. Schmeidler (1969) introduced this point-valued solution concept (see Dreissen (1988)). The nucleolus of supply chain game v is

$$\left(N(v)_m = \frac{J_{ms}^c + J_m^d - J_s^d}{2}, N(v)_s = \frac{J_{ms}^c + J_s^d - J_m^d}{2} \right).$$

The nucleolus can be interpreted as it is such an allocation that minimizes the maximal exceeds the coalitions can achieve. It is a slight calculation to see that in our model the nucleolus and the Shapley value coincide. This, the following lemma is about.

Lemma 7 The nucleolus and the Shapley solution coincide in supply chain games, i.e. for any supply chain game v $N(v) = Sh(v)$.

Moreover, Lemma 5 implies that the nucleolus of supply chain games is in the stable set, i.e. for any supply chain game v $N(v) \in S(v)$. It is well known that the nucleolus is always in the core, if the core is nonempty; therefore that the core of a supply chain game is not empty and Lemma 7 imply Lemma 5.

5 A numerical example

Take the following parameters and cost functions in problems (1)-(2), (3)-(4) and (5)-(8):

- the initial inventory level of the manufacturer: $I_{m0} = 0.5$,
- the initial inventory level of the supplier: $I_{s0} = 0.3$,
- the planning horizon: $T = 5$ years,
- the demand rate of the manufacturer: $S(t) = 0.45 \cdot t^2$,
- the inventory holding cost of the manufacturer: $h_m = 2$,
- the inventory holding cost of the supplier: $h_s = 1$,
- the production cost of the manufacturer: $F_m(P_m(t)) = 0.5 \cdot P_m^2(t)$,
- the production cost of the supplier: $F_s(P_s(t)) = 5 \cdot P_s^2(t)$.

In the following we solve the decentralized and the centralized problem.

5.1 The solution of the decentralized problem

The decentralized problem is a hierarchical production planning problem. First the manufacturer solves her planning problem then the optimal ordering policy is forwarded to the supplier. Finally, the supplier optimizes her own relevant costs based on the known ordering policy of the manufacturer.

The problem of the manufacturer is as follows:

$$\int_0^5 \left[2 \cdot I_m(t) + 0.5 \cdot P_m^2(t)\right] dt \to min$$

s.t.

$$\dot{I}_m(t) = P_m(t) - 0.5 \cdot t^2,\ I_m(0) = 0.5, \quad 0 \le t \le 5$$

The optimal solution is

$$P_m^d(t)=\begin{cases} 0, & if\ 0\le t<0.728,\\ 2\cdot t-1.456, & if\ t\le 0.728\le 5, \end{cases}$$

and

$$I_m^d(t)=\begin{cases} 0.5-0.15\cdot t^3, & if\ 0\le t<0.728,\\ 1.03-1.456\cdot t+t^2-0.15\cdot t^3, & if\ t\le 0.728\le 5. \end{cases}$$

The minimal cost of the manufacturer is 62.078 units.

In the next step we solve the problem of the supplier, where the manufacturer's ordering policy $P_m^d(\cdot)$ is given:

$$\int_0^5 \left[1\cdot I_s(t)+5\cdot P_s^2(t)\right]dt\to \min$$

s.t.

$$\dot{I}_s(t)=P_s(t)-P_m^d(t),\ I_s(0)=I_{s0},\quad 0\le t\le 5$$

The optimal solution for the supplier is

$$P_s^d(t)=0.334+0.1\cdot t,\ t\in[0,5]$$

and

$$I_s^d(t)=\begin{cases} 0.3+0.05\cdot t^3, & if\ 0\le t<0.728,\\ -0.23+4.796\cdot t-0.95t^2, & if\ t\le 0.728\le 5. \end{cases}$$

The minimal cost of the supplier is 342.096 units.

5.2 The solution of the centralized problem

In the following we solve the centralized problem:

$$\int_0^T \left[2\cdot I_m(t)+0.5\cdot P_m^2(t)+1\cdot I_s(t)+5\cdot P_s^2(t)\right]dt\to min$$

s.t.

$$\dot{I}_m(t)=P_m(t)-S(t),\quad 0\le t\le 5$$

$$\dot{I}_s(t)=P_s(t)-P_m(t),\quad 0\le t\le 5$$

$$\begin{pmatrix} I_m(0)\\ I_s(0)\end{pmatrix}=\begin{pmatrix}0.5\\0.3\end{pmatrix}$$

The optimal production rates are the followings:

$$P_m^c(t) = 1.15 + t, t \in [0,5]$$

and

$$P_s^c(t) = 3.09 + 0.2 \cdot t, t \in [0,5].$$

The optimal inventory levels for the manufacturer and the supplier respectively are

$$I_m^c(t) = 0.5 + 1.15 \cdot t + 0.5 \cdot t^2 - 0.15 \cdot t^3, \quad t \in [0,5]$$

and

$$I_s^c(t) = 0.3 + 1.94 \cdot t - 0.4 \cdot t^2, \ t \in [0,5].$$

The minimal cost of the centralized system is 400.425 units, where the manufacturer's cost is 67.056 units and the supplier's cost is 333.369 units.

5.3 Comparison of the solutions of the decentralized and the centralized system

First, compare the production rate and inventory level of the manufacturer and the supplier in the cases of the decentralized and the centralized system, where Imd(t), Imc(t), Isd(t) and Isc(t) are for the inventory level for the manufacturer and for the supplier in the decentralized and the centralized model respectively.

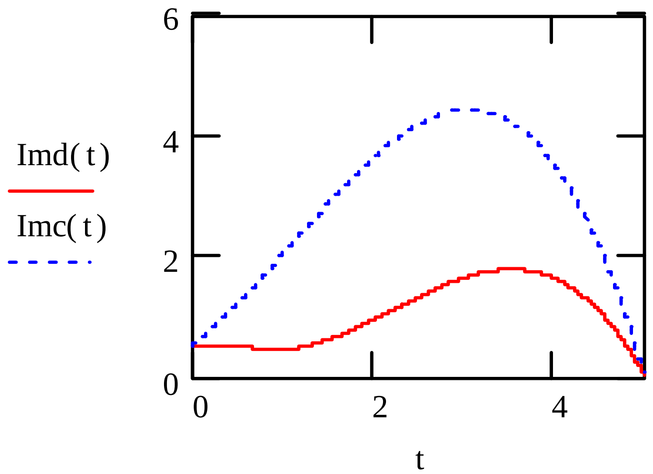

Figure 2 The inventory level of the manufacturer in the decentralized and the centralized system

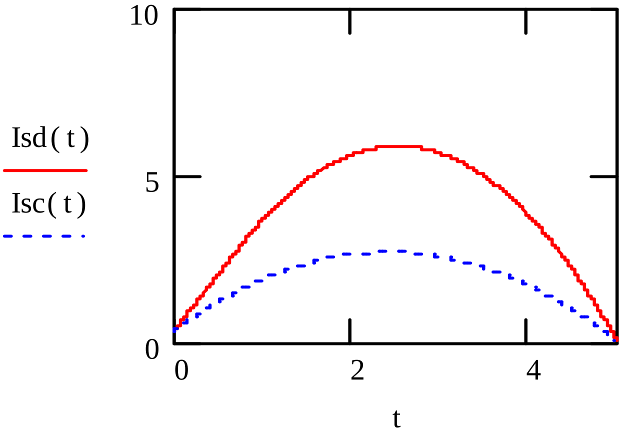

Figure 3 The inventory level of the supplier in the decentralized and the centralized system

In this example the inventory level of the manufacturer decreases in the case of cooperation, i.e. in the centralized system. The inventory level of the supplier increases when the participants cooperate in the supply chain, see Figures 2 and 3.

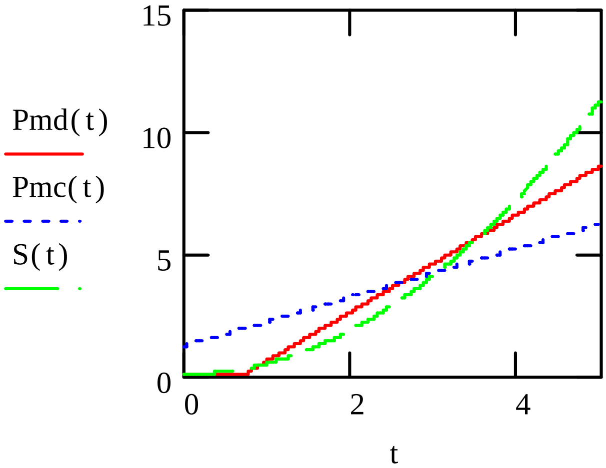

Figure 4 The production rate of the manufacturer in the decentralized and the centralized system

As we see, the production level in the centralized system is smoother, i.e. the growth of the production rate is smaller than that in the case of the decentralized system, and the contrary is true for the supplier, i.e. in the decentralized system the production rate of the supplier is smoother than that in the centralized system, where Pmd(t), Pmc(t), Psd(t) and Psc(t) are for the production level for the manufacturer and for the supplier in the decentralized and the centralized models respectively, and S(t) is for the exogenously given demand, see Figures 4 and 5. This phenomenon is the decreased bullwhip effect in the centralized model.

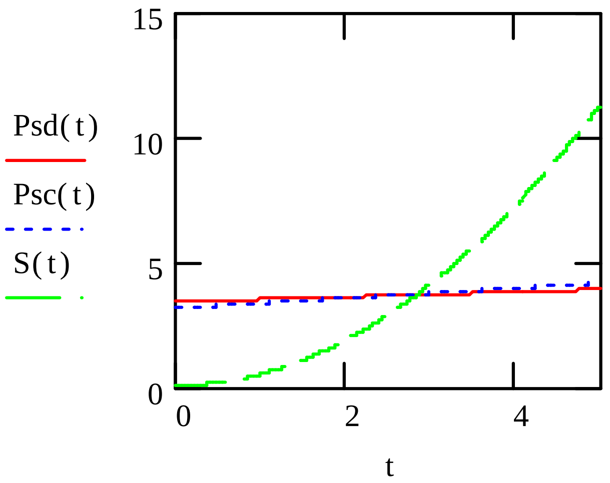

Figure 5 The production rate of manufacturer in the decentralized and the centralized system

The optimal costs of the decentralized and the centralized problem are presented in Table 1.

	Decentralized problem	**Centralized Problem**
Manufacturer costs	62.078	67.056
Supplier costs	342.069	333.369
Total costs	404.148	400.425

Table 1 The optimal costs

As we have seen, the total cost of the centralized problem is lower than that of the decentralized one. The cost reduction is approximately 1%. In the centralized problem the manufacturer cost increases with more than 8% and the supplier cost decreases with 2.5%.

After the above analysis the question of how to share the savings, the cooperation of the participants in the supply chain induces, comes up.

5.4 Cost sharing

The Shapley value of the manufacturer and the supplier (it coincides with the nucleolus and is in the core and in the stable set) are $Sh_m(v) = 60.217$ and $Sh_s(v) = 340.208$ respectively. It means that the players share their savings equally.

It is important to see that since in the case of cooperation $J_m^c = 67.056$ and $J_s^c = 333.369$ a transfer is needed to get the Shapley value: the supplier must transfer 6.839 units to the manufacturer. It means that the manufacturer and the supplier agree on a contract such that the parties commit themselves to cooperate and the supplier commits herself to pay 6.839 units to the manufacturer.

6. Conclusion and further research

In this paper we have solved two two-stage supply chain models: a decentralized and a centralized model. We have showed that the cooperation of the two players induces savings in costs.

In the next step we have considered sharing rules for the savings. We have applied cooperative game theory solution concepts to this problem, and we have introduced the concept of supply chain games. It was shown that in supply chain games the core and the stable set coincide, so do the Shapley value and the nucleolus; therefore the Shapley value is always in the core.

As an illustration for our results we have presented an exact number example. In this example the supplier's cost of adaption in production to the fluctuations in the orderings of the manufacturer is higher than that of the manufacturer. Moreover, the production costs are dominant over the inventory costs. Therefore it is not surprising at all that in the centralized model the supplier has reduced her inventory level, and the manufacturer's inventory level is higher than that in the decentralized model, and vice versa for the supplier.

The reason of this fact is that the manufacturer minimizes her relevant cost in the decentralized model, so that her production level is near to the demand rate. After cooperation the manufacturer gives up to follow her cost optimal production strategy to allow the supplier to reduce her own production-inventory cost implying a decrease in the total cost of the supply chain as well, since the supplier's cost saving balances out the increase of the manufacturer's cost.

This phenomenon points at the well known bullwhip effect of supply chains in a way: the supplier decreased the inventory level after information sharing (cooperation), and she adjusted her production rate closer to the demand rate.

In this type supply chains the two players might have asymmetrical roles. It can happen that the manufacturer has much stronger bargaining position than that of the supplier or vice versa. Since this asymmetry in the bargaining powers is exogenously given, it is not reflected by the proposed solution, by the Shapley value. The future research can propose solutions concepts which can reflect the exogenously given bargaining powers.

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